The symmetrising trace conjecture for Hecke algebras
(joint work with C. Boura, E. Chavli & K. Karvounis)

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**Theorem (Shephard–Todd)**

Let $W \subset GL(V)$ be an irreducible complex reflection group (i.e., $W$ acts irreducibly on $V$). Then one of the following assertions is true:

- $W \cong G(de, e, r)$, where $G(de, e, r)$ is the group of all $r \times r$ monomial matrices whose non-zero entries are $de$-th roots of unity, while the product of all non-zero entries is a $d$-th root of unity.

- $W \cong G_n$ for some $n = 4, \ldots, 37$. 

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- We have $Z(B(W)) = \langle \zeta_W \rangle$. In the examples, $\zeta_{G_4} = ststst$ and $\zeta_{G_5} = stst$.
- We have $B(W) \twoheadrightarrow \mathcal{H}(W)$, $\zeta_W \mapsto z_W$. 
The Broué–Malle–Rouquier freeness conjecture
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**Theorem (since October)**

The algebra $\mathcal{H}(W)$ is a free $R_W$-module of rank $|W|$.

It has been proved for:

- the real reflection groups by Bourbaki;
- the complex reflection groups $G(de, e, r)$ by Ariki–Koike, Broué–Malle, Ariki;
- the group $G_4$ by Broué–Malle, Funar, Marin;
- the group $G_{12}$ by Marin–Pfeiffer;
- the groups $G_4, \ldots, G_{16}$ by Chavli;
- the groups $G_{17}, G_{18}, G_{19}$ by Tsuchioka;
- the groups $G_{20}, G_{21}$ by Marin;
- the groups $G_{22}, \ldots, G_{37}$ by Marin, Marin–Pfeiffer.
The Broué–Malle–Michel symmetrising trace conjecture

Let $B$ be an $R_W$-basis for $H(W)$. Conjecture

There exists a linear map $\tau : H(W) \to R_W$ that satisfies the following conditions:

1. $\tau$ is a symmetrising trace, that is, the matrix $A := (\tau(b_i b_j) b_i, b_j \in B)$ is symmetric and invertible over $R_W$.

2. When $H(W)$ specialises to the group algebra of $W$, $\tau$ becomes the canonical symmetrising trace given by $\tau(w) = \delta_1 w$ for all $w \in W$.

3. $\tau$ satisfies an extra condition, which makes it unique.

It has been proved for:

- the real reflection groups by Bourbaki;
- the complex reflection groups $G(\text{de}, e, r)$ by Bremke–Malle, Malle–Mathas;
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The idea of the algorithm for $G_4, \ldots, G_8$

We have $|G_4| = 24$, $|G_5| = 72$, $|G_6| = 48$, $|G_7| = 144$, $|G_8| = 96$.

**STEP 1:** Let $n \in \{4, \ldots, 8\}$. Take a basis $B_n$ for each $H(G_n)$ and define a linear map $\tau$ on $H(G_n)$ by setting $\tau(b) := \delta_1 b$ for all $b \in B_n$. We must have $1 \in B_n$ and $B_n = W$ when $H(W)$ specialises to the group algebra of $W$.

By construction, $B_n$ satisfies the second condition of the BMM symmetrising trace conjecture.

If $h \in H(G_n)$, then $\tau(h)$ is the coefficient of 1 when $h$ is expressed as a linear combination of the elements of $B_n$.

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In her proof of the BMR freeness conjecture, Chavli provided explicit bases for $H(G_n)$ for $n = 4, \ldots, 16$. However, note that not any basis will work for the proof of the BMM symmetrising trace conjecture! If not, go back to **STEP 1** and modify $B_n$ accordingly.

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**STEP 2:** Calculate the matrix $A = (\tau(b_i b_j)_{b_i, b_j \in B_n})$. Check whether $A$ is symmetric and invertible over $R_W$. If yes, then $\tau$ satisfies the first condition of the BMM symmetrising trace conjecture.

In her proof of the BMR freeness conjecture, Chavli provided explicit bases for $\mathcal{H}(G_n)$ for $n = 4, \ldots, 16$. However, note that not any basis will work for the proof of the BMM symmetrising trace conjecture!

If not, go back to **STEP 1** and modify $B_n$ accordingly.

**STEP 3:** Check that the extra third condition holds.
The C++ algorithm

For any $b_i, b_j \in B_n$, our C++ program expresses $b_i b_j$ as a linear combination of the elements of $B_n$. 

Running the C++ program takes about 1 hour on an Intel Core i5 CPU.
The C++ algorithm

For any \( b_i, b_j \in B_n \), our C++ program expresses \( b_i b_j \) as a linear combination of the elements of \( B_n \). Then \( \tau(b_i b_j) \) is the coefficient of 1 in this linear combination.
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The inputs of the algorithm are the following:

I1. The basis \( B_n \).

I2. The braid, positive and inverse Hecke relations (for example, \( s^{-1} = c^{-1}s^2 - ac^{-1}s - bc^{-1} \)).

I3. The “special cases”: these are some equalities computed by hand which express a given element of \( \mathcal{H}(G_n) \) as a sum of other elements in \( \mathcal{H}(G_n) \).
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I3. The “special cases”: these are some equalities computed by hand which express a given element of $H(G_n)$ as a sum of other elements in $H(G_n)$.

The case of $G_4$

We have $B_4 = \{1, s, s^2, t^2, t, t^2 s, ts, t^2 s^2, ts^2, st^2, st, st^2 s, sts, st^2 s^2, sts^2, s^2 t^2, s^2 t, s^2 t^2 s, s^2 ts, s^2 t^2 s^2, s^2 ts^2, stst, ststst, stststst\}$.

Running the C++ program takes about 1 hour on an Intel Core i5 CPU.
The SAGE algorithm

Our SAGE program produces the matrix $A$ row by row, using the distinctive pattern of the basis $B_n$.

There exists a set $E_n$ with $1, s, t \in E_n$ such that $B_n = \{ z_k e | e \in E_n, k = 0, 1, \ldots, |Z(G_n)| - 1 \}$. We have $E_n \leftrightarrow G_n / Z(G_n) \sim \{ A_4 \text{ for } n = 5, 6, 7; S_4 \text{ for } n = 8 \}$.

The curious case of $G_7$: 3 elements have to be replaced!

The inputs of the SAGE algorithm are the coefficients of the following elements when written as linear combinations of the elements of $B_n$:

I1. $s b_j$ for all $b_j \in B_n$.

I2. $t b_j$ for all $b_j \in B_n$.

I3. $z | Z(G_n) | = z \cdot z | Z(G_n) | - 1$. 
The SAGE algorithm

Our SAGE program produces the matrix $A$ row by row, using the distinctive pattern of the basis $B_n$.

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We have $E_n \leftrightarrow G_n/Z(G_n) \cong \begin{cases} \mathfrak{A}_4 & \text{for } n = 5, 6, 7; \\ \mathfrak{S}_4 & \text{for } n = 8. \end{cases}$
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Our SAGE program produces the matrix $A$ row by row, using the distinctive pattern of the basis $B_n$.

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The case of $G_5$

- $\mathcal{H}(G_5) = \langle s, t \mid stst = tsts, \ s^3 = as^2 + bs + c, \ t^3 = dt^2 + et + f \rangle$. 
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We set $b_{12k+m} := z^k b_m$. We observe that we have:

\[
\begin{align*}
b_{12k+2} &= b_{12k+1} \cdot s, & b_{12k+8} &= b_{12k+6} \cdot t, \\
b_{12k+3} &= b_{12k+2} \cdot s, & b_{12k+9} &= b_{12k+7} \cdot t, \\
b_{12k+4} &= b_{12k+1} \cdot t, & b_{12k+10} &= f^{-1}(b_{12k+5} - db_{12k+4} - eb_{12k+1}) \cdot s, \\
b_{12k+5} &= b_{12k+4} \cdot t, & b_{12k+11} &= b_{12k+10} \cdot t, \\
b_{12k+6} &= b_{12k+2} \cdot t, & b_{12k+12} &= b_{12k+11} \cdot t. \\
b_{12k+7} &= b_{12k+3} \cdot t
\end{align*}
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The case of $G_5$

Let $j \in \{1, \ldots, 72\}$. Using the C++ program, we have expressed $sb_j$, $tb_j$ and $z^6 = b_{37}^2$ as linear combinations of the elements of $B_5$ with coefficients in $\mathbb{Z}[a, b, c^{\pm 1}, d, e, f^{\pm 1}]$. 
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Examples

- $\tau(b_{12k+4} b_j) = \tau(b_{12k+1} tb_j) = \sum_l \lambda_{j,l}^t \tau(b_{12k+1} b_l)$. 
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- \( \tau(b_{12k+10}b_j) = f^{-1} \sum_l \lambda_{j,l}^s (\tau(b_{12k+5}b_l) - d \tau(b_{12k+4}b_l) - e \tau(b_{12k+1}b_l)) \).
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We now consider the case of $b_{12k+1} = z^k$, for $k \neq 0$. We distinguish two cases:
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- If $1 \leq j \leq 12(6 - k)$, then we have $b_{12k+1}b_j \in B_5$, whence $\tau(b_{12k+1}b_j) = 0$. 
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$$\tau(b_{12k+1} b_j) = \tau(b_{12k+j-72} \cdot z^6) = \sum_l \mu_l \tau(b_{12k+j-72} b_l).$$
The GAP algorithm

Let $W$ be a complex reflection group. Under the BMR freeness conjecture, Malle has shown that there exists a field $K$ over which $\mathcal{H}(W)$ is split semisimple.
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$$\tau = \sum_{\chi \in \text{Irr}(K\mathcal{H}(W))} \frac{1}{s_{\chi}} \chi$$

where $s_{\chi} \in K$ denotes the Schur element of $K\mathcal{H}(W)$ associated with $\chi$. 

STEP 1: Define $\tilde{\tau}$ as the RHS of (1).

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Our program worked for $G_4$ and $G_6$. It produced $A$ for $G_8$, but could not calculate $\det(A)$. It could not even establish STEP 2 for $G_5$ and $G_7$. 
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Schur elements have been completely determined by Malle for all non-real exceptional complex reflection groups.
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The extra condition

Malle and Michel have shown that, since

1. each element of $B_n$ corresponds to a distinct element of $G_n$
   (that is, $B_n = \{ T_w \mid w \in G_n \}$),

\[ \tau(b) = \delta_1 b \text{ for all } b \in B_n, \]

and $B_n$ is a basis of $H(G_n)$ as an $R_{G_n}$-module,

the extra condition of the BMM symmetrising trace conjecture translates as:

\[ \tau(z \mid Z(G_n) \mid b - 1) = 0 \text{ for all } b \in B_n \setminus \{1\}. \]  

(2)

This is equivalent to

\[ \sum_{\chi \in \text{Irr}(KH(G_n))} \omega_{\chi}(z \mid Z(G_n)) s_{\chi}(\chi(b - 1)) = 0 \text{ for all } b \in B_n \setminus \{1\}. \]  

(3)

We used GAP to prove Formula (3) for $G_4$, $G_6$, and $G_8$.

We directly proved Formula (2) for $G_5$ and $G_7$, by expressing $\tau(z \mid Z(G_n) \mid b - 1)$ as a linear combination of entries of the matrix $A$. 

\[ A \]
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2. \( \tau(b) = \delta_{1b} \) for all \( b \in B_n \), and

This is equivalent to

\[
\sum_{\chi \in \text{Irr}(KH(G_n))} \omega_{\chi}(z|Z(G_n)|b - 1) s_{\chi}\chi(b - 1) = 0 \quad \text{for all } b \in B_n \backslash \{1\}.
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- We directly proved Formula (2) for $G_5$ and $G_7$, by expressing $\tau \left( z^{\mid Z(G_n) \mid} b^{-1} \right)$ as a linear combination of entries of the matrix $A$. 
The main results

Theorem (Boura–Chavli–C.–Karvounis)

Let $n \in \{4, \ldots, 8\}$. The BMM symmetrising trace conjecture holds for $G_n$. 

Our C++ program has expressed $s_b$ and $t_b$ as linear combinations of the elements of $B_n$, for all $b_j \in B_n$ (in the case of $G_7$, the product of the third generator with any $b_j$ is deduced from the other two). This in fact allows us to express any product of the generators, and thus any element, of $H(G_n)$ as a linear combination of the elements of $B_n$. 

Theorem (Boura–Chavli–C.–Karvounis)

Let $n \in \{4, \ldots, 8\}$. The set $B_n$ is a basis for $H(G_n)$ as an $R_{G_n}$-module. In particular, the BMR freeness conjecture holds for $G_n$. 

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Let \( n \in \{4, \ldots, 8\} \). The set \( B_n \) is a basis for \( \mathcal{H}(G_n) \) as an \( R_{G_n} \)-module. In particular, the BMR freeness conjecture holds for \( G_n \).