The symmetrising trace conjecture for Hecke algebras

(joint work with C. Boura, E. Chavli & K. Karvounis)

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Let $W \subset \mathrm{GL}(V)$ be an irreducible complex reflection group (i.e., W acts irreducibly on V). Then one of the following assertions is true:

- $W \cong G(de, e, r)$, where $G(de, e, r)$ is the group of all $r \times r$ monomial matrices whose non-zero entries are de-th roots of unity, while the product of all non-zero entries is a d-th root of unity.
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We define the rank of W to be the dimension of V.

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- We have $B(W) \twoheadrightarrow \mathcal{H}(W),\ \zeta_W \mapsto z_W$.

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- the complex reflection groups $G(de, e, r)$ by Ariki–Koike, Broué–Malle, Ariki;
- the group G_4 by Broué–Malle, Funar, Marin;
- the group G_{12} by Marin–Pfeiffer;
- the groups $G_4, \ldots, G_{16}$ by Chavli;
- the groups G_{17}, G_{18}, G_{19} by Tsuchioka;
- the groups G_{20}, G_{21} by Marin;
- the groups $G_{22}, \ldots, G_{37}$ by Marin, Marin–Pfeiffer.

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- the real reflection groups by Bourbaki;
- the complex reflection groups $G(de, e, r)$ by Bremke–Malle, Malle–Mathas;
- the groups G_4, G_{12}, G_{22}, G_{24} by Malle–Michel (G_4 also by Marin–Wagner).

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The inputs of the algorithm are the following:

I1. The basis $\mathcal{B}_n$.

I2. The braid, positive and inverse Hecke relations (for example, $s^{-1} = c^{-1}s^2 - ac^{-1}s - bc^{-1}$).

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The case of G_4

We have $\mathcal{B}_4 = \left\{ \begin{array}{l} 1, s, s^2, t^2, t, t^2s, ts, t^2s^2, ts^2, st^2, st, st^2s, sts, st^2s^2, sts^2, \\ s^2t^2, s^2t, s^2t^2s, s^2ts, s^2t^2s^2, s^2ts^2, ststst,ststs, ststs^2 \end{array} \right\}$.

Running the C++ program takes about 1 hour on an Intel Core i5 CPU.

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The curious case of G_7

3 elements have to be replaced!

The SAGE algorithm

Our SAGE program produces the matrix A row by row, using the distinctive pattern of the basis $\mathcal{B}_n$.

Set $z := z_{G_n}$. There exists a set $\mathcal{E}_n$ with $1, s, t \in \mathcal{E}_n$ such that

$$\mathcal{B}_n = \{z^k e \mid e \in \mathcal{E}_n,\ k = 0, 1, \ldots, |Z(G_n)| - 1\}.$$

We have $\mathcal{E}_n \leftrightarrow G_n/Z(G_n) \cong \begin{cases} \mathfrak{A}_4 & \text{for } n = 5, 6, 7; \\ \mathfrak{S}_4 & \text{for } n = 8. \end{cases}$

The curious case of G_7

3 elements have to be replaced!

The inputs of the SAGE algorithm are the coefficients of the following elements when written as linear combinations of the elements of $\mathcal{B}_n$:

I1. sb_j for all $b_j \in \mathcal{B}_n$.

I2. tb_j for all $b_j \in \mathcal{B}_n$.

I3. $z^{|Z(G_n)|} = z \cdot z^{|Z(G_n)|-1}$.

The case of G_5

- $\mathcal{H}(G_5) = \left\langle s,\ t \;\middle|\; stst = tsts,\ \ s^3 = as^2 + bs + c,\ \ t^3 = dt^2 + et + f \right\rangle .$

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b_{12k+2} &= b_{12k+1} \cdot s, & b_{12k+8} &= b_{12k+6} \cdot t,\\
b_{12k+3} &= b_{12k+2} \cdot s, & b_{12k+9} &= b_{12k+7} \cdot t,\\
b_{12k+4} &= b_{12k+1} \cdot t, & b_{12k+10} &= f^{-1}(b_{12k+5} - db_{12k+4} - eb_{12k+1}) \cdot s,\\
b_{12k+5} &= b_{12k+4} \cdot t, & b_{12k+11} &= b_{12k+10} \cdot t,\\
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Let $j \in \{1, \dots, 72\}$. Using the C++ program, we have expressed sb_j, tb_j and $z^6 = b_{37}^2$ as linear combinations of the elements of $\mathcal{B}_5$ with coefficients in $\mathbb{Z}[a, b, c^{\pm 1}, d, e, f^{\pm 1}]$.

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Our program worked for G_4 and G_6. It produced A for G_8, but could not calculate $\det(A)$. It could not even establish STEP 2 for G_5 and G_7.

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- We directly proved Formula (2) for G_5 and G_7, by expressing $\tau\left(z^{|Z(G_n)|} b^{-1}\right)$ as a linear combination of entries of the matrix A.

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Let $n \in \{4, \ldots, 8\}$. The set $\mathcal{B}_n$ is a basis for $\mathcal{H}(G_n)$ as an R_{G_n}-module. In particular, the BMR freeness conjecture holds for G_n.