Blocks of generic and cyclotomic Hecke algebras

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Example: *If $\mathcal{O} = \mathbb{Z}$ and $A = \mathbb{Z}[G]$ (G a finite group), we can define the following symmetrizing form ("canonical") on A*

$$t : \mathbb{Z}[G] \to \mathbb{Z},\ \sum_{g \in G} a_g g \mapsto a_1.$$

Theorem (Geck) - Definition

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3. Let L be a field such that LA is split. If $\theta : \mathcal{O}_K \to L$ is a ring homomorphism, then LA is semisimple if and only if $\theta(s_\chi) \neq 0$ for all $\chi \in \mathrm{Irr}(KA)$

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$$\mathbf{u} = (u_{s,j})_{s,\,0 \leq j \leq \mathbf{o}(s)-1}$$

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Remark: The specialization $u_{s,j} \mapsto \zeta_{\mathbf{o}(s)}^j$ sends $\mathcal{H}(W)$ to $\mathbb{Z}_K W$.

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Theorem (Malle)

Let $\mathbf{v} = (v_{s,j})_{s,j}$ be a set of indeterminates such that, for all s, j, we have

$$v_{s,j}^{|\mu(K)|} := \zeta_{\mathbf{o}(s)}^{-j} u_{s,j},$$

where $\mu(K)$ is the group of all the roots of unity in K. Then the $K(\mathbf{v})$-algebra $K(\mathbf{v})\mathcal{H}$ is split semisimple.

By "Tits' deformation theorem", the specialization $v_{s,j} \mapsto 1$ induces a bijection

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Let $\chi \in \mathrm{Irr}(W)$. The Schur element $s_{\chi_{\mathbf{v}}}$ is an element of $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ whose irreducible factors (in $K[\mathbf{v}, \mathbf{v}^{-1}]$) are of the form: $\Psi(M)$

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Schur elements of G_2 : $X_0^2 := u_0$, $X_1^2 := -u_1$, $Y_0^2 := w_0$, $Y_1^2 := -w_1$. 2-essential in purple, 3-essential in green.

$$s_1 = \Phi_4(X_0X_1^{-1}) \cdot \Phi_4(Y_0Y_1^{-1}) \cdot \Phi_3(X_0Y_0X_1^{-1}Y_1^{-1}) \cdot \Phi_6(X_0Y_0X_1^{-1}Y_1^{-1})$$

$$s_2 = 2 \cdot X_1^2X_0^{-2} \cdot \Phi_3(X_0Y_0X_1^{-1}Y_1^{-1}) \cdot \Phi_6(X_0Y_1X_1^{-1}Y_0^{-1})$$

$\Phi_4(x) = x^2+1$, $\Phi_3(x) = x^2+x+1$, $\Phi_6(x) = x^2-x+1$

$\Phi_4(1) = 2$ $\Phi_3(1) = 3$ $\Phi_6(1) = 1$

Cyclotomic Hecke algebras

Let y be an indeterminate. A cyclotomic specialization of $\mathcal{H}$ is a $\mathbb{Z}_K$-algebra morphism $\phi : \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ of the form:

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Proposition (C.)

The algebra $K(y)\mathcal{H}_\phi$ is split semisimple.

By "Tits' deformation theorem", we obtain that the specialization $v_{s,j} \mapsto 1$ induces the following bijections :

$$\begin{array}{ccccc} \mathrm{Irr}(K(\mathbf{v})\mathcal{H}) & \leftrightarrow & \mathrm{Irr}(K(y)\mathcal{H}_\phi) & \leftrightarrow & \mathrm{Irr}(W) \\ \chi_{\mathbf{v}} & \mapsto & \chi_\phi & \mapsto & \chi \end{array}$$

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Proposition

The Schur element s_{χ_ϕ} associated to the irreducible character χ_ϕ of $K(y)\mathcal{H}_\phi$ is a Laurent polynomial in y of the form

$$s_{\chi_\phi} = \psi_{\chi,\phi} \cdot y^{a_{\chi,\phi}} \cdot \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi,\phi}},$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

Rouquier blocks, $\mathfrak{p}$-blocks and $\mathfrak{p}$-essential monomials

The Rouquier blocks of the cyclotomic Hecke algebra $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi$, where

$$\mathcal{R}_K(y) := \mathbb{Z}_K[y, y^{-1}, (y^n - 1)^{-1}_{n \geq 1}]$$

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Proposition

Set $\mathcal{O} := \mathbb{Z}_K[y, y^{-1}]$. Two irreducible characters $\chi, \psi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$ belong to the same Rouquier block if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$ and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of prime ideals of $\mathbb{Z}_K$ such that

- $\chi_0 = \chi$ and $\chi_n = \psi$,
- forall i $(1 \leq i \leq n)$, χ_{i-1} and χ_i belong to the same block of $\mathcal{O}_{\mathfrak{p}_i\mathcal{O}}\mathcal{H}_\phi$.

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Theorem (C.)

Two irreducible characters $\chi, \psi \in \mathrm{Irr}(W)$ are in the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{q}_{j_1}, \ldots, \mathfrak{q}_{j_n} \in \mathcal{Q}$ such that

- $\chi_0 = \chi$ and $\chi_n = \psi$,
- for all i $(1 \leq i \leq n)$, χ_{i-1} and χ_i belong to the same block of $\mathcal{A}_{\mathfrak{q}_{j_i}}\mathcal{H}$.