Workshop on Representation Theory
Lefkosia, Cyprus

Blocks and families for cyclotomic Hecke algebras

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The blocks of A are the primitive idempotents of the center ZA of A.

Suppose that there exists a finite Galois extension K of F such that the algebra $KA := K \otimes_{\mathcal{O}} A$ is split semisimple.

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If $\chi \in B$ for $B \in \mathrm{Bl}(A)$, we say that "χ belongs to the block e_B".

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Lemma

A trace function $t : A \to \mathcal{O}$ is symmetrizing if and only if there exist two bases $(e_1, \ldots, e_n)$ and $(e'_1, \ldots, e'_n)$ of A over $\mathcal{O}$ such that $t(e_i e'_j) = \delta_{ij}$.

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Corollary

The blocks of A are the non-empty subsets B of $\mathrm{Irr}(KA)$ which are minimal with respect to the property:

$$\sum_{\chi \in B} \frac{\chi(a)}{s_\chi} \in \mathcal{O}, \text{ for all } a \in A.$$

Example: If $\mathcal{O} = \mathbb{Z}$ and $A = \mathbb{Z}G$ (G a finite group), we can define the following symmetrizing form ("canonical symmetrizing form") on A

$$t : \mathbb{Z}[G] \to \mathbb{Z}, \ \sum_{g \in G} a_g g \mapsto a_1.$$

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For all $\chi \in \mathrm{Irr}(G)$, we have

$$s_\chi = \frac{|G|}{\chi(1)}.$$

Real reflection groups

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Theorem

A finite group W is a real reflection group if and only if W is a Coxeter group.

Hecke algebras

Let (W, S) be a finite Coxeter system. Then W has a presentation of the form:

$$W = \langle\ S \mid \underbrace{ststst\ldots}_{m(s,t)} = \underbrace{tststs\ldots}_{m(s,t)},\ s^2 = 1,\ \forall s,\ t \in S\ \rangle$$

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The generic Hecke algebra $\mathcal{H}(W)$ of W is defined over the Laurent polynomial ring $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$, where $\mathbf{u} = (u_{s,0}, u_{s,1})_{s \in S}$ is a set of indeterminates, and has a presentation of the form:

$$\mathcal{H}(W) = \langle\, (T_s)_{s \in S} \mid \underbrace{T_s T_t T_s \ldots}_{m(s,t)} = \underbrace{T_t T_s T_t \ldots}_{m(s,t)},\ (T_s - u_{s,0}^2)(T_s + u_{s,1}^2) = 0,\ \forall s,\ t \in S\,\rangle$$

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We also ask that $u_{s,j}=u_{t,j}$ whenever s and t are conjugate in W.

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Moreover, the algebra $\mathbb{Q}(\mathbf{u})\mathcal{H}(W)$ is split semisimple. By "Tits' deformation theorem", the specialization $u_{s,j} \mapsto 1$ induces a bijection

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We have

$$t=\sum_{\chi\in\mathrm{Irr}(W)}\frac{1}{s_{\chi_{\mathbf{u}}}}\chi_{\mathbf{u}}.$$

Let q be an indeterminate. A cyclotomic specialization of $\mathcal{H}(W)$ is a $\mathbb{Z}$-algebra morphism $\varphi : \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}] \rightarrow \mathbb{Z}[q, q^{-1}]$ of the form:

$$\varphi : u_{s,j} \mapsto q^{n_{s,j}}, \quad \text{where } n_{s,j} \in \mathbb{Z} \text{ for all } s \text{ and } j.$$

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Families of characters and Rouquier blocks

1. The families of characters of a Weyl group W, defined by Lusztig, are a partition of the set of irreducible characters of W which plays a key-role in the organization of the families of unipotent characters of the corresponding finite reductive group. (cf. [Lusztig, 1984]).

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2. The families of characters are defined with the help of the Kazhdan-Lusztig basis of the Iwahori-Hecke algebra of W. In fact, the families are determined by the two-sided cells of W. As a consequence, the definition of families can be applied to all finite Coxeter groups.

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3. More recent results of Gyoja (1996) and Rouquier (1999) have given us a substitute for the definition of the familles. In particular, Rouquier has shown that the families of characters of the group W coincide with the blocks of the classical Iwahori-Hecke algebra of W over a suitable coefficient ring, the Rouquier ring.

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$$\sum_{\chi \in B} \frac{\chi_\varphi(h)}{s_{\chi_\varphi}} \in \mathcal{R}_{\mathbb{Q}}(q), \text{ for all } h \in \mathcal{H}_\varphi.$$

Theorem

- The Schur element $s_{\chi_{\mathbf{u}}}$ of $\chi_{\mathbf{u}} \in \mathrm{Irr}(\mathbb{Q}(\mathbf{u})\mathcal{H}(W))$ is an element of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ of the form:

$$s_{\chi_{\mathbf{u}}} = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i}),$$

where

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- The Schur element s_{χ_φ} of $\chi_\varphi \in \mathrm{Irr}(\mathbb{Q}(q)\mathcal{H}_\varphi)$ is of the form:

$$s_{\chi_\varphi} = \psi_{\chi_\varphi} q^{a_{\chi_\varphi}} \prod_{\Phi \in C_{\mathbb{Q}}} \Phi(q),$$

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The hyperplane $\sum_{s,j} a_{s,j} t_{s,j} = 0$ is an essential hyperplane for W.

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Examples:

$$\mathcal{H}(G_2) = \left\langle\; T_s, T_t \;\middle|\; \begin{array}{l} T_sT_tT_sT_tT_sT_t = T_tT_sT_tT_sT_tT_s, \\ (T_s - u_0^2)(T_s + u_1^2) = (T_t - v_0^2)(T_t + v_1^2) = 0 \end{array} \;\right\rangle$$

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Schur elements of G_2 (essential in green):

$$s_1 = \Phi_4(u_0u_1^{-1}) \cdot \Phi_4(v_0v_1^{-1}) \cdot \Phi_3(u_0u_1^{-1}v_0v_1^{-1}) \cdot \Phi_6(u_0u_1^{-1}v_0v_1^{-1})$$

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Let $\varphi : u_0 \mapsto q^{m_0},\ u_1 \mapsto q^{m_1},\ v_0 \mapsto q^{n_0},\ v_1 \mapsto q^{n_1}$ be a cyclotomic specialization. The essential hyperplanes for G_2 are:

$$M_0 = M_1,\ N_0 = N_1,\ M_0 - M_1 = N_0 - N_1,\ M_0 - M_1 = N_1 - N_0.$$

Essential Hyperplane H	$\mathcal{BR}_H(G_2)$
$\emptyset$	$(\chi_{1,0}),\ (\chi_{1,6}),\ (\chi_{1,3'}),\ (\chi_{1,3''}),\ (\chi_{2,1},\chi_{2,2})$
$M_0 = M_1$	$(\chi_{1,0},\chi_{1,3'}),\ (\chi_{1,6},\chi_{1,3''}),\ (\chi_{2,1},\chi_{2,2}),$
$N_0 = N_1$	$(\chi_{1,0},\chi_{1,3''}),\ (\chi_{1,6},\chi_{1,3'}),\ (\chi_{2,1},\chi_{2,2}),$
$M_0 - M_1 = N_1 - N_0$	$(\chi_{1,3'}),\ (\chi_{1,3''}),\ (\chi_{1,0},\chi_{1,6},\chi_{2,1},\chi_{2,2})$
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Let us take $m_0 := 1$, $m_1 := 0$, $n_0 := 1$ and $n_1 := 0$.

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Let us take $m_0 := 1$, $m_1 := 0$, $n_0 := 1$ and $n_1 := 0$. These integers belong only to the essential hyperplane $M_0 - M_1 = N_0 - N_1$. Therefore, the Rouquier blocks of $\mathcal{H}(G_2)_\varphi$ are:

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$$\mathcal{H}(\mathfrak{S}_n) = \left\langle \; T_1, T_2, \ldots, T_{n-1} \;\middle|\; \begin{array}{l} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i \text{ if } |i-j| > 1, \\ (T_i - u_0^2)(T_i + u_1^2) = 0 \end{array} \right\rangle$$

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Essential Hyperplane H	$\mathcal{BR}_H(\mathfrak{S}_n)$
$\emptyset$	All characters alone
$M_0 = M_1$	All characters together

Complex reflection groups

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Let W be an irreducible complex reflection group. Then W is isomorphic to

- either the group $G(de, e, r)$, where $d, e, r \in \mathbb{Z}^+$,
- or one of the exceptional groups G_n $(n = 4, \ldots, 37)$.

1. The complex reflection groups and their associated cyclotomic Hecke algebras appear naturally in the classification of the "cyclotomic Harish-Chandra series" of the characters of the finite reductive groups, generalizing the role of the Weyl group and its traditional Hecke algebra in the principal series (cf. [Broué, Malle, Michel, 1993], [Broué, Malle, 1993]). Since the families of characters of the Weyl group play an essential role in the definition of the families of unipotent characters of the corresponding finite reductive group, we can hope that the families of characters of the cyclotomic Hecke algebras play a key role in the organization of families of unipotent characters more generally.

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2. For some complex reflection groups (non-Coxeter), some data have been gathered which seem to indicate that behind the group W, there exists another mysterious object — the *Spets* — that could play the role of the "series of finite reductive groups with Weyl group W" (cf. [Broué, Malle, Michel, 1999]). In some cases, one can define the unipotent characters of the Spets, which are controlled by the "spetsial" Hecke algebra of W, a generalization of the classical Iwahori-Hecke algebra of the Weyl groups.

Assumptions (verified for all but a finite number of cases)

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- The generic Hecke algebra $\mathcal{H}(W)$ is a free $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-module of rank $|W|$.
- There exists a canonical symmetrizing form on $\mathcal{H}(W)$.

The example of G_4

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$$\mathcal{H}(G_4) = \left\langle\ T_s, T_t \ \middle|\ \begin{array}{l} T_s T_t T_s = T_t T_s T_t, \\ (T_s - u_0)(T_s - \zeta_3 u_1)(T_s - \zeta_3^2 u_2) = 0, \\ (T_t - u_0)(T_t - \zeta_3 u_1)(T_t - \zeta_3^2 u_2) = 0 \end{array} \right\rangle$$

The algebra $\mathcal{H}(G_4)$ is defined over $\mathbb{Z}[\zeta_3][u_0, u_1, u_2, u_0^{-1}, u_1^{-1}, u_2^{-1}]$, where $\zeta_3 = \exp(2\pi i/3)$.

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We denote the characters of G_4 by: $\chi_{1,0}, \chi_{1,4}, \chi_{1,8}, \chi_{2,5}, \chi_{2,3}, \chi_{2,1}, \chi_{3,2}$.

Schur elements of G_4 (essential in green):

$$s_{1,0} = \Phi_2(u_0 u_1^{-1}) \cdot \Phi_3'(u_0 u_1^{-1}) \cdot \Phi_6'(u_0 u_1^{-1}) \cdot \Phi_2(u_0 u_2^{-1}) \cdot \Phi_3''(u_0 u_2^{-1}) \cdot \Phi_6''(u_0 u_2^{-1}) \cdot \Phi_2(u_0^2 u_1^{-1} u_2^{-1})$$

$$s_{2,1} = -\zeta_3^2 u_0^{-1} u_1 \cdot \Phi_2(u_0 u_1^{-1}) \cdot \Phi_6'(u_0 u_1^{-1}) \cdot \Phi_3''(u_0 u_2^{-1}) \cdot \Phi_3'(u_1 u_2^{-1}) \cdot \Phi_2(u_0 u_1 u_2^{-2})$$

$$s_{3,2} = \Phi_2(u_0^{-2} u_1 u_2) \cdot \Phi_2(u_0 u_1^{-2} u_2) \cdot \Phi_2(u_0 u_1 u_2^{-2})$$

$\Phi_2(x) = x+1$, $\Phi_3'(x) = x-\zeta_3$, $\Phi_3''(x) = x-\zeta_3^2$, $\Phi_6'(x) = x+\zeta_3^2$, $\Phi_6''(x) = x+\zeta_3$.

Schur elements of G_4 (essential in green):

$$s_{1,0} = \Phi_2(u_0u_1^{-1}) \cdot \Phi'_3(u_0u_1^{-1}) \cdot \Phi'_6(u_0u_1^{-1}) \cdot \Phi_2(u_0u_2^{-1}) \cdot \Phi''_3(u_0u_2^{-1}) \cdot \Phi''_6(u_0u_2^{-1}) \cdot \Phi_2(u_0^2u_1^{-1}u_2^{-1})$$

$$s_{2,1} = -\zeta_3^2u_0^{-1}u_1 \cdot \Phi_2(u_0u_1^{-1}) \cdot \Phi'_6(u_0u_1^{-1}) \cdot \Phi''_3(u_0u_2^{-1}) \cdot \Phi'_3(u_1u_2^{-1}) \cdot \Phi_2(u_0u_1u_2^{-2})$$

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Let $\varphi : u_0 \mapsto q^{m_0}$, $u_1 \mapsto q^{m_1}$, $u_2 \mapsto q^{m_2}$ be a cyclotomic specialization of $\mathcal{H}(G_4)$.

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Let $\varphi : u_0 \mapsto q^{m_0}$, $u_1 \mapsto q^{m_1}$, $u_2 \mapsto q^{m_2}$ be a cyclotomic specialization of $\mathcal{H}(G_4)$. The essential hyperplanes for G_2 are:

$$M_0 = M_1,\ M_0 = M_2,\ M_1 = M_2,$$

$$2M_0 = M_1 + M_2,\ 2M_1 = M_0 + M_2,\ 2M_2 = M_0 + M_1$$

Essential Hyperplane H	$\mathcal{BR}_H(G_4)$
$\emptyset$	$(\chi_{1,0}), (\chi_{1,4}), (\chi_{1,8}), (\chi_{2,5}), (\chi_{2,3}), (\chi_{2,1}), (\chi_{3,2})$
$M_0 = M_1$	$(\chi_{1,8}), (\chi_{1,0}, \chi_{1,4}, \chi_{2,1}), (\chi_{2,5}, \chi_{2,3}), (\chi_{3,2})$
$M_0 = M_2$	$(\chi_{1,4}), (\chi_{1,0}, \chi_{1,8}, \chi_{2,3}), (\chi_{2,5}, \chi_{2,1}), (\chi_{3,2})$
$M_1 = M_2$	$(\chi_{1,0}), (\chi_{1,4}, \chi_{1,8}, \chi_{2,5}), (\chi_{2,3}, \chi_{2,1}), (\chi_{3,2})$
$2M_0 = M_1 + M_2$	$(\chi_{1,0}, \chi_{2,5}, \chi_{3,2}), (\chi_{1,4}), (\chi_{1,8}), (\chi_{2,3}), (\chi_{2,1})$
$2M_1 = M_0 + M_2$	$(\chi_{1,4}, \chi_{2,3}, \chi_{3,2}), (\chi_{1,0}), (\chi_{1,8}), (\chi_{2,5}), (\chi_{2,1})$
$2M_2 = M_0 + M_1$	$(\chi_{1,8}, \chi_{2,1}, \chi_{3,2}), (\chi_{1,0}), (\chi_{1,4}), (\chi_{2,5}), (\chi_{2,3})$

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$M_0 = M_1$	$(\chi_{1,8}), (\chi_{1,0}, \chi_{1,4}, \chi_{2,1}), (\chi_{2,5}, \chi_{2,3}), (\chi_{3,2})$
$M_0 = M_2$	$(\chi_{1,4}), (\chi_{1,0}, \chi_{1,8}, \chi_{2,3}), (\chi_{2,5}, \chi_{2,1}), (\chi_{3,2})$
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$2M_0 = M_1 + M_2$	$(\chi_{1,0}, \chi_{2,5}, \chi_{3,2}), (\chi_{1,4}), (\chi_{1,8}), (\chi_{2,3}), (\chi_{2,1})$
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$2M_2 = M_0 + M_1$	$(\chi_{1,8}, \chi_{2,1}, \chi_{3,2}), (\chi_{1,0}), (\chi_{1,4}), (\chi_{2,5}), (\chi_{2,3})$

Let us take $m_0 := 1$, $m_1 := 0$ and $m_2 := 0$.

Essential Hyperplane H	$\mathcal{BR}_H(G_4)$
$\emptyset$	$(\chi_{1,0})$, $(\chi_{1,4})$, $(\chi_{1,8})$, $(\chi_{2,5})$, $(\chi_{2,3})$, $(\chi_{2,1})$, $(\chi_{3,2})$
$M_0 = M_1$	$(\chi_{1,8})$, $(\chi_{1,0}, \chi_{1,4}, \chi_{2,1})$, $(\chi_{2,5}, \chi_{2,3})$, $(\chi_{3,2})$
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$M_1 = M_2$	$(\chi_{1,0})$, $(\chi_{1,4}, \chi_{1,8}, \chi_{2,5})$, $(\chi_{2,3}, \chi_{2,1})$, $(\chi_{3,2})$
$2M_0 = M_1 + M_2$	$(\chi_{1,0}, \chi_{2,5}, \chi_{3,2})$, $(\chi_{1,4})$, $(\chi_{1,8})$, $(\chi_{2,3})$, $(\chi_{2,1})$
$2M_1 = M_0 + M_2$	$(\chi_{1,4}, \chi_{2,3}, \chi_{3,2})$, $(\chi_{1,0})$, $(\chi_{1,8})$, $(\chi_{2,5})$, $(\chi_{2,1})$
$2M_2 = M_0 + M_1$	$(\chi_{1,8}, \chi_{2,1}, \chi_{3,2})$, $(\chi_{1,0})$, $(\chi_{1,4})$, $(\chi_{2,5})$, $(\chi_{2,3})$

Let us take $m_0 := 1$, $m_1 := 0$ and $m_2 := 0$. These integers belong only to the essential hyperplane $M_1 = M_2$.

Essential Hyperplane H	$\mathcal{BR}_H(G_4)$
$\emptyset$	$(\chi_{1,0})$, $(\chi_{1,4})$, $(\chi_{1,8})$, $(\chi_{2,5})$, $(\chi_{2,3})$, $(\chi_{2,1})$, $(\chi_{3,2})$
$M_0 = M_1$	$(\chi_{1,8})$, $(\chi_{1,0}, \chi_{1,4}, \chi_{2,1})$, $(\chi_{2,5}, \chi_{2,3})$, $(\chi_{3,2})$
$M_0 = M_2$	$(\chi_{1,4})$, $(\chi_{1,0}, \chi_{1,8}, \chi_{2,3})$, $(\chi_{2,5}, \chi_{2,1})$, $(\chi_{3,2})$
$M_1 = M_2$	$(\chi_{1,0})$, $(\chi_{1,4}, \chi_{1,8}, \chi_{2,5})$, $(\chi_{2,3}, \chi_{2,1})$, $(\chi_{3,2})$
$2M_0 = M_1 + M_2$	$(\chi_{1,0}, \chi_{2,5}, \chi_{3,2})$, $(\chi_{1,4})$, $(\chi_{1,8})$, $(\chi_{2,3})$, $(\chi_{2,1})$
$2M_1 = M_0 + M_2$	$(\chi_{1,4}, \chi_{2,3}, \chi_{3,2})$, $(\chi_{1,0})$, $(\chi_{1,8})$, $(\chi_{2,5})$, $(\chi_{2,1})$
$2M_2 = M_0 + M_1$	$(\chi_{1,8}, \chi_{2,1}, \chi_{3,2})$, $(\chi_{1,0})$, $(\chi_{1,4})$, $(\chi_{2,5})$, $(\chi_{2,3})$

Let us take $m_0 := 1$, $m_1 := 0$ and $m_2 := 0$. These integers belong only to the essential hyperplane $M_1 = M_2$. Therefore, the Rouquier blocks of $\mathcal{H}(G_4)_\varphi$ are:

$$(\chi_{1,0}),\ (\chi_{1,4}, \chi_{1,8}, \chi_{2,5}),\ (\chi_{2,3}, \chi_{2,1}),\ (\chi_{3,2}).$$