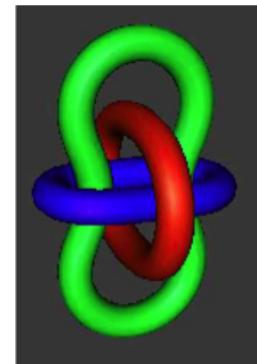
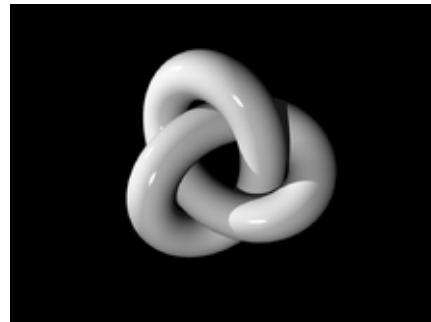


YOKONUMA-HECKE ALGEBRAS & KNOT INVARIANTS

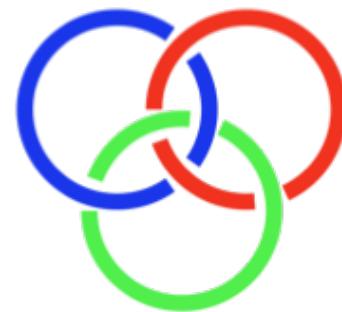
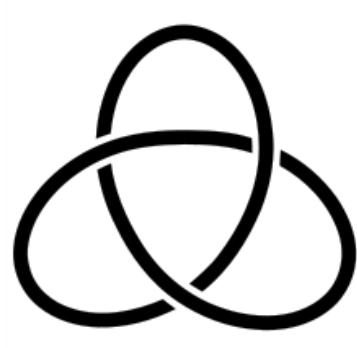
Maria Chlouveraki

Université de Versailles - St Quentin

A **knot** (respectively a **link**) is an embedding of the circle S^1 (resp. n copies of S^1) into 3-dimensional Euclidean space \mathbb{R}^3 .



Every link can be represented by a "knot diagram"



\mathcal{L} = set of links

A knot invariant is a function $I : \mathcal{L} \rightarrow S$ (S a set)

such that

$$L_1 \sim L_2 \implies I(L_1) = I(L_2)$$

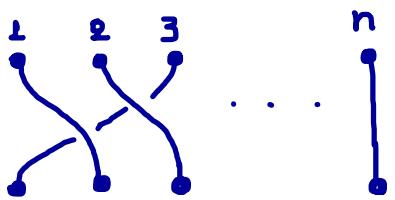
for $L_1, L_2 \in \mathcal{L}$.

Braid group (of type A)

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i = 1, \dots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| > 1 \end{array} \right\rangle$$

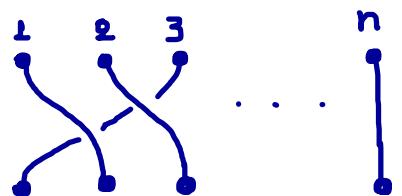
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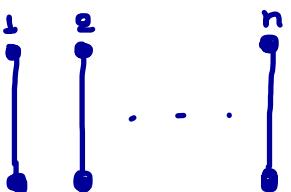
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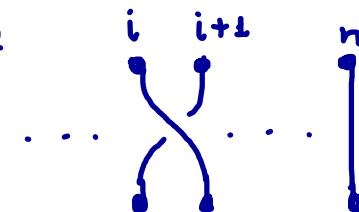


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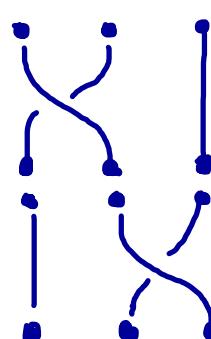
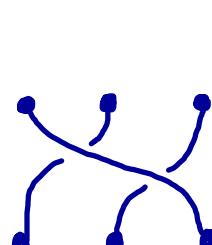
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Identity = 

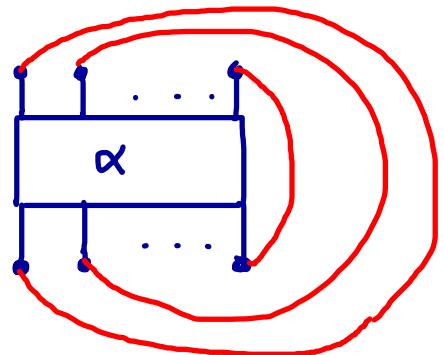
$\sigma_i =$ 

Multiplication : concatenation of diagrams

E.g. $\alpha = \sigma_1 =$  $\Rightarrow \alpha \beta = \sigma_1 \sigma_2 =$ 

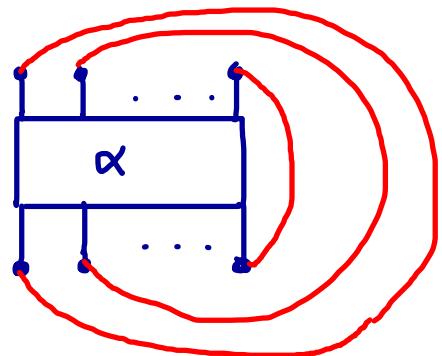
$\beta = \sigma_2 =$ 

Every element of B_n gives rise to a knot or link



$= : \hat{\alpha} = \text{closure of } \alpha$

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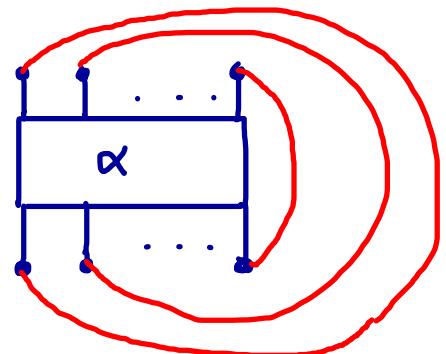
E.g. $\alpha =$ $\hat{\alpha} =$

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$$\sim$$

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$$\alpha = \sigma_1^3$$
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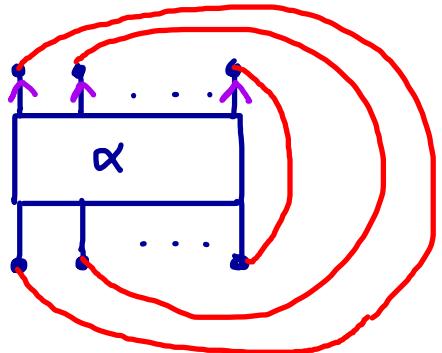
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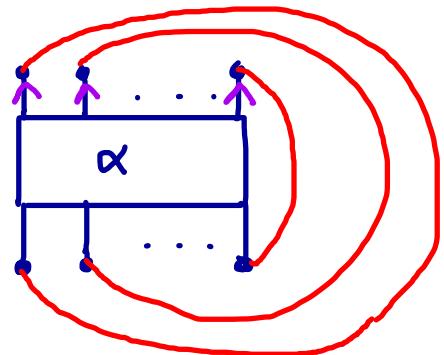
$$\alpha = \sigma_i^3$$
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Every element of B_n gives rise to an oriented knot or link



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Every (oriented) link can be obtained as the closure $\hat{\alpha}$ of a braid $\alpha \in \bigcup_{n \geq 1} B_n$.

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We define an equivalence relation on $\bigcup_{n \geq 1} B_n$ as the transitive closure of the relations :

- (i) $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in B_n$ (conjugation)
- (ii) $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n$ (Markov's move)

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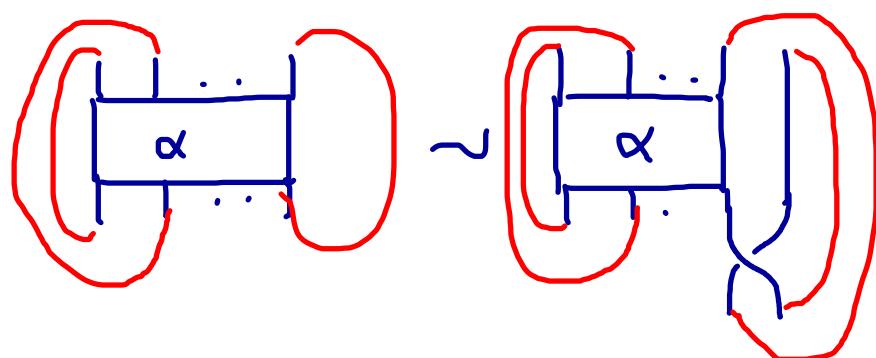
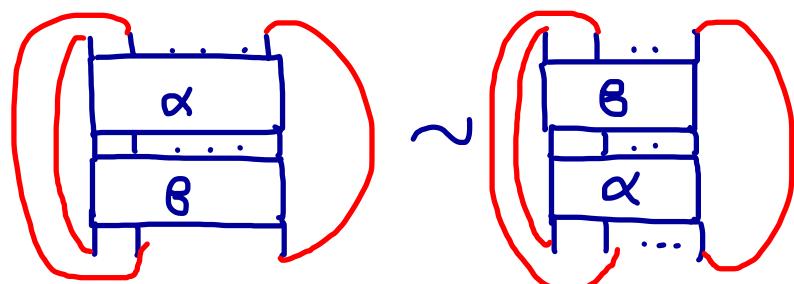
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Markov's Theorem (1935)

We have $\hat{\alpha} \sim \hat{\beta}$ if and only if $\alpha \sim \beta$.



An invariant of oriented knots & links is a function $I : \bigcup_{n \geq 1} B_n \longrightarrow S$ such that

- (i) $I(\alpha\beta) = I(\beta\alpha)$ $\forall \alpha, \beta \in B_n$
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Here S will be a set of Laurent polynomials.

Iwahori - Hecke algebra of type A

q indeterminate , $R = \mathbb{C}[q, q^{-1}]$

$$\mathfrak{H}_n(q) = \left\{ G_1, \dots, G_{n-1} \right\} \quad \begin{array}{l} G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} \\ G_i G_j = G_j G_i \quad \text{if } |i-j| > 1 \\ G_i^q = (q-1) G_i + q \end{array}$$

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Normalisation of $\tau \rightarrow$ HOMFLYPT (or 2-variable Jones) polynomial

Framed braid group

Let $d \in \mathbb{Z}_{>0}$

$$\begin{array}{c} (\mathbb{Z}/d\mathbb{Z})^l \times B_n \\ \parallel \\ (\mathbb{Z}/d\mathbb{Z})^n \times B_n \end{array} = \left\langle \sigma_1, \dots, \sigma_{n-1}, t_1, \dots, t_n \right| \begin{array}{l} \sigma_i \text{ as before} \\ t_j^d = 1 \quad j=1, \dots, n \\ t_i t_j = t_j t_i \quad i, j=1, \dots, n \\ t_j \sigma_i = \sigma_i t_{s_i(j)} \end{array}$$

where $s_i = (i, i+1) \in S_n$

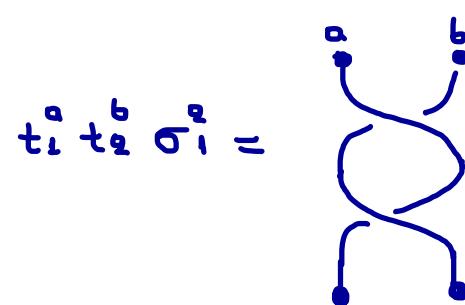
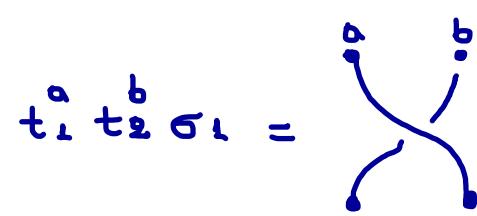
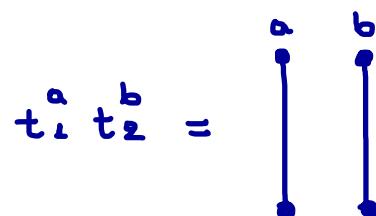
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E.g. $a, b \in \{0, 1, \dots, d-1\}$, $n=2$



Multiplication : concatenation of diagrams

$(t_1^a t_2^b \sigma_1) (t_1^{a'} t_2^{b'}) = t_1^{a+b'} t_2^{b+a'} \sigma_1$

Every element of $(\mathbb{Z}/d\mathbb{Z}) \wr B_n$ gives rise to a **framed** knot or link

E.g. $a, b \in \{0, 1, \dots, d-1\}$, $n=2$

$$t_1^a t_2^b \sigma_1 = \text{Diagram} \xrightarrow{\sim} \text{Link } a+b$$

$$t_1^a t_2^b \sigma_1^2 = \text{Diagram} \xrightarrow{\sim} \text{Link } a \text{ } b$$

$$d=3 : \quad \overbrace{t_1 t_2 \sigma_1} \sim \overbrace{t_1^2 \sigma_1}, \quad \overbrace{t_1 t_2 \sigma_1^2} \not\sim \overbrace{t_1^2 \sigma_1^2}$$

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$$t_1^a t_2^b \sigma_1^2 = \text{Diagram} \rightarrow \text{Two circles } a \text{ and } b$$

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Alexander's Theorem: obvious

Markov's Theorem : Ko - Smolinsky 1992

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- $d = 1$: $Y_{1,n}(q) \cong \mathcal{H}_n(q)$
- $\mathcal{H}_n(q)$ is a quotient of $Y_{d,n}(q)$ ($t_j \mapsto 1$)

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Normalisation of tr



E-condition on $(x_0, x_1, \dots, x_{d-1})$

Juyumaya-Lambropoulou
invariant for framed knots and links

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“Forget” the framings

Invariant for classical
knots and links

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Proposition [C.-Lambropoulou]

If $E_D = 1$, then $\Delta_{d,D}$ coincides with the HOMFLYPT polynomial
 $(= \Delta_{L,\{0\}})$

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- The invariants Δ_d can be defined via a skein relation applied in 2 steps.
- If we take $E := \text{tr}(e_i)$ to be an indeterminate, we can define a 3-variable invariant with all the above properties.

Representation theory of $\mathrm{Y}_{d,n}(q)$

- Thiem 2005 : Unipotent Hecke algebras
- C.-Poulain d'Andecy : Jucys-Murphy elements
Explicit combinatorial formulas
Semisimplicity criterion / Schur elements

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$$\text{Irr}(\mathbb{C}(q)Y_{d,n}(q)) \leftrightarrow \text{Irr}(G(d,1,n)) \leftrightarrow \{ \text{d-partitions of } n \}$$

A **d-partition of n** is a family of d partitions

$\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)})$ such that $|\lambda^{(1)}| + |\lambda^{(2)}| + \dots + |\lambda^{(d)}| = n$.

$$\mathbb{C}[G(d, 1, n)]$$



Yokonuma - Hecke algebra

- Wreath product
- Braid group of type A
- Repⁿ theory of $\mathfrak{sl}_n(q)$



Ariki - Koike algebra

- Quadratic relation for g_i
- Braid group of type B
- $\mathfrak{sl}_n(q)$ (obvious) subalgebra

Cyclotomic Yokonuma-Hecke algebra $Y(d,m,n)$

[C. - Poulain d'Andecy]

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$\xrightarrow{\sim}$
[CJKL]

Stronger than the above

Temperley-Lieb algebra

Let $n \geq 3$.

$$TL_n(q) := \mathfrak{sl}_n(q) / \langle 1 + G_1 + G_2 + G_1G_2 + G_2G_1 + G_1G_2G_1 \rangle$$

Temperley-Lieb algebra

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$$\text{Irr}(\mathbb{C}(q)\mathfrak{U}_n(q)) \leftrightarrow \{\text{partitions of } n\}$$

$$\text{Irr}(\mathbb{C}(q)TL_n(q)) \leftrightarrow \left\{ \begin{array}{l} \text{partitions of } n \text{ whose Young diagrams} \\ \text{have at most 2 columns} \end{array} \right\}$$

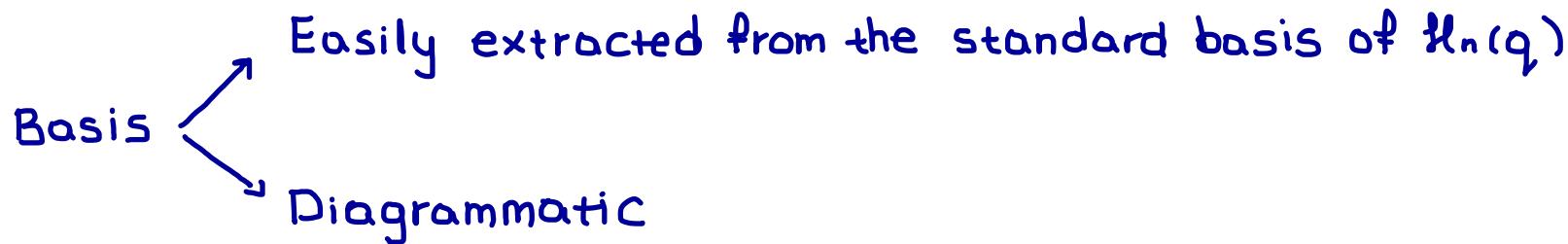
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Yokohama-Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geq 3$.

$$YTL_{d,n}(q) := Y_{d,n}(q) / \langle 1 + g_1 + g_2 + g_1g_2 + g_2g_1 + g_1g_2g_1 \rangle$$

Yokonuma-Temperley-Lieb algebra

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[C.-Pouchin]

$$\text{Irr}(\mathbb{C}(q)YTL_{d,n}(q)) \longleftrightarrow \left\{ \begin{array}{l} d\text{-partitions } \gamma = (\gamma^{(1)}, \dots, \gamma^{(d)}) \text{ of } n \\ \text{such that the Young diagrams of} \\ \text{all } \gamma^{(i)} \text{ together have at most 2 columns} \end{array} \right\}$$

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Basis : extracted from the standard basis of $Y_{d,n}(q)$ (with difficulty !)

Framisation of the Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geq 3$.

$$FTL_{d,n}(q) := Y_{d,n}(q) / \langle e_1 e_2 (1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1) \rangle$$

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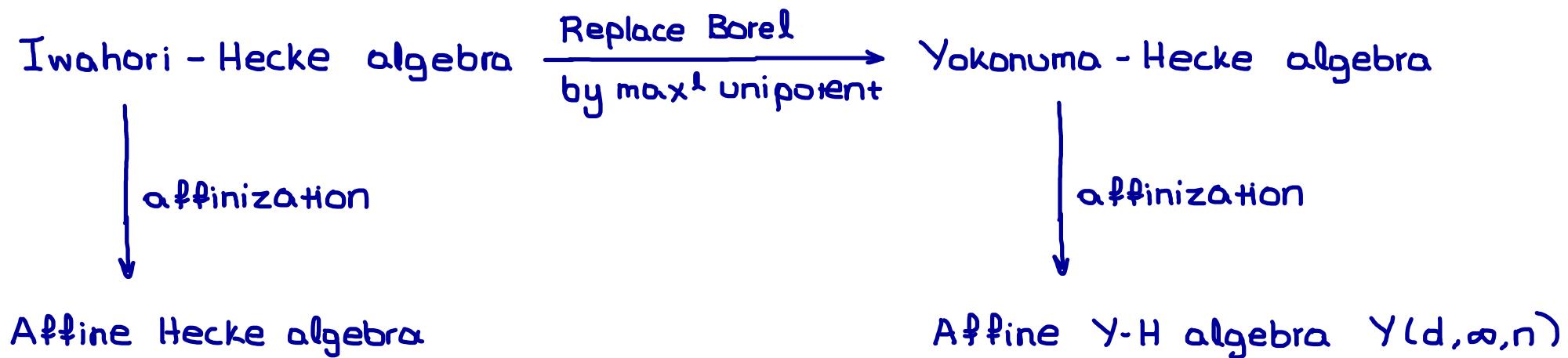
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Basis : Work in progress!

The affine Yokonuma-Hecke algebra $Y(d, \infty, n)$

[C.-Poulain d'Andecy]



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