Decomposition matrices for cyclotomic Hecke algebras

Maria Chlouveraki

University of Edinburgh

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Complex reflection groups

A complex reflection group W is a finite group of matrices with coefficients in a finite abelian extension K of \mathbb{Q} generated by *pseudo-reflections*.

If $K = \mathbb{Q}$, then W is a Weyl group.

Shephard-Todd classification (1954)

The irreducible complex reflection groups are:

• the groups of the infinite series G(de, e, r)

(with $G(d, 1, r) \cong \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_r$);

• the exceptional groups G_4, G_5, \ldots, G_{37} .

Hecke algebras of complex reflection groups

Let W be a complex reflection group.

The group W has a presentation given by:

- generators: S
- relations:
 - braid relations;

•
$$(s-1)(s-\zeta_{e_s})\cdots(s-\zeta_{e_s})=0.$$
 $[\zeta_{e_s}:=\exp(2\pi i/e_s)]$

Example:

$$G := G(3,1,2) = \langle s,t | stst = tsts, s^3 = 1, t^2 = 1 \rangle.$$

Let q be an indeterminate and let $A := \mathbb{Z}_{K}[q, q^{-1}]$.

The cyclotomic Hecke algebra $\mathcal{H}_q(W)$ has a presentation given by:

- generators: $(T_s)_{s\in S}$
- relations:
 - braid relations;

•
$$(T_s - 1q^{m_{s,0}})(T_s - \zeta_{e_s}q^{m_{s,1}})\cdots(T_s - \zeta_{e_s}^{e_s - 1}q^{m_{s,e_s - 1}}) = 0.$$

Example: G = G(3, 1, 2)

$$\mathcal{H}_{q}(G) = \left\langle T_{s}, T_{t} \middle| \begin{array}{c} T_{s}T_{t}T_{s}T_{t} = T_{t}T_{s}T_{t}T_{s}, \\ (T_{s} - q^{m_{s,0}})(T_{s} - \zeta_{3}q^{m_{s,1}})(T_{s} - \zeta_{3}^{2}q^{m_{s,2}}) = 0, \end{array} \right\rangle.$$

$$(T_{t} - q^{m_{t,0}})(T_{t} + q^{m_{t,1}}) = 0$$

Schur elements of Hecke algebras

(We make some assumptions.)

The algebra $K(q)\mathcal{H}_q(W)$ is semisimple. By Tits's deformation theorem, we have a bijection:

$$\frac{\operatorname{Irr}(\mathcal{K}(q)\mathcal{H}_q(W))}{\chi_q} \stackrel{\leftrightarrow}{\mapsto} \frac{\operatorname{Irr}(W)}{\chi}.$$

Moreover, there exists a "canonical" symmetrizing form $t : \mathcal{H}_q(W) \to A$, such that

$$t = \sum_{\chi \in \operatorname{Irr}(W)} \frac{1}{s_{\chi}} \chi_q$$

where s_{χ} is the Schur element of $\mathcal{H}_q(W)$ associated with χ .

We have that s_{χ} belongs to $A = \mathbb{Z}_{K}[q, q^{-1}]$ and it is a product of cyclotomic polynomials over K.

Definition

We define a_{χ} to be the smallest non-negative integer such that

 $q^{a_{\chi}}s_{\chi} \in \mathbb{Z}_{K}[q].$

Example: If $s_{\chi} = q^{-1} + 2 + q$, then $a_{\chi} = 1$.

The decomposition matrix

Let

$$\theta: A \to \mathbb{C}, \quad q \mapsto \xi$$

be a ring homomorphism. Set $\mathcal{H}_{\xi} := \mathbb{C} \otimes_{A} \mathcal{H}_{q}(W)$.

Theorem (Geck-Pfeiffer)

The algebra \mathcal{H}_{ξ} is semisimple if and only if $\theta(s_{\chi}) \neq 0$ for all $\chi \in Irr(W)$.

We have a well-defined decomposition map

$$d_{ heta}: R_0(K(q)\mathcal{H}_q(W)) o R_0(\mathcal{H}_{\xi})$$

with corresponding decomposition matrix

$$D_{\theta} = ([E:M])_{E \in \operatorname{Irr}(W), M \in \operatorname{Irr}(\mathcal{H}_{\xi})}.$$

Basic sets

Definition (Geck-Rouquier)

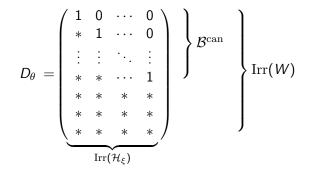
We say that $\mathcal{H}_q(W)$ admits a canonical basic set $\mathcal{B}^{\operatorname{can}} \subset \operatorname{Irr}(W)$ with respect to $\theta : A \to \mathbb{C}$ if there exists a bijection

$Irr(\mathcal{H}_{\xi})$	\leftrightarrow	$\mathcal{B}^{ ext{can}}$
М	\mapsto	Eм

such that

- $[E_M : M] = 1$, and
- if $[E:M] \neq 0$, then either $E = E_M$ or $a_E > a_{E_M}$.

If $\mathcal{H}_q(W)$ admits a canonical basic set \mathcal{B}^{can} with respect to θ , then the decomposition matrix D_{θ} has the following form:



Theorem

The algebra $\mathcal{H}_q(W)$ admits a canonical basic set with respect to any specialization $\theta: A \to \mathbb{C}$, if

- W is a Weyl group; [Geck-Rouquier, Geck, Geck-Jacon, C.-Jacon]
- W is a complex reflection group of type G(d, 1, r);
 [Dipper-James-Murphy, Geck-Rouquier, Ariki, Uglov, Jacon]
- W is a complex reflection group of type G(de, e, r) (for a certain choice of paremeters); [Genet-Jacon]
- W ∈ {G₄, G₅, G₈, G₉, G₁₀, G₁₂, G₁₆, G₂₀, G₂₂} (for certain choices of paremeters). [C.-Miyachi]

In the last case, we have been also able to show that there exists a subset $\mathcal{B}^{\text{opt}} \subset \text{Irr}(W)$ such that the decomposition matrix D_{θ} has the following form:

