Decomposition matrices for cyclotomic Hecke algebras

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A complex reflection group $W$ is a finite group of matrices with coefficients in a finite abelian extension $K$ of $\mathbb{Q}$ generated by pseudo-reflections.

If $K = \mathbb{Q}$, then $W$ is a Weyl group.

Shephard-Todd classification (1954)

The irreducible complex reflection groups are:

- the groups of the infinite series $G(de, e, r)$
  
  (with $G(d, 1, r) \cong \mathbb{Z}/d\mathbb{Z} \wr S_r$);

- the exceptional groups $G_4$, $G_5$, $\ldots$, $G_{37}$. 
Hecke algebras of complex reflection groups

Let $W$ be a complex reflection group.

The group $W$ has a presentation given by:

- generators: $S$
- relations:
  - braid relations;
  - $(s - 1)(s - \zeta_e) \cdots (s - \zeta_e^{-1}) = 0$.  
    
    \(
    \zeta_e := \exp(2\pi i/e_s)
    \)

Example:

\[
G := G(3, 1, 2) = \langle s, t \mid stst = tsts, s^3 = 1, t^2 = 1 \rangle.
\]
Let \( q \) be an indeterminate and let \( A := \mathbb{Z}_K[q, q^{-1}] \).

The cyclotomic Hecke algebra \( \mathcal{H}_q(W) \) has a presentation given by:

- **generators:** \( (T_s)_{s \in S} \)
- **relations:**
  - braid relations;
  - \((T_s - 1q^{m_s,0})(T_s - \zeta_{e_s} q^{m_s,1}) \cdots (T_s - \zeta_{e_s}^{-1} q^{m_s, e_s - 1}) = 0\).

**Example:** \( G = G(3,1,2) \)

\[
\mathcal{H}_q(G) = \left\langle T_s, T_t \right| \begin{array}{l}
T_s T_t T_s T_t = T_t T_s T_t T_s, \\
(T_s - q^{m_s,0})(T_s - \zeta_3 q^{m_s,1})(T_s - \zeta_3^2 q^{m_s,2}) = 0, \\
(T_t - q^{m_t,0})(T_t + q^{m_t,1}) = 0
\end{array} \right\}.
\]
Schur elements of Hecke algebras

(We make some assumptions.)

The algebra $K(q)\mathcal{H}_q(W)$ is semisimple. By Tits’s deformation theorem, we have a bijection:

$$\text{Irr}(K(q)\mathcal{H}_q(W)) \leftrightarrow \text{Irr}(W)$$

$\chi_q \mapsto \chi$.

Moreover, there exists a “canonical” symmetrizing form $t : \mathcal{H}_q(W) \to A$, such that

$$t = \sum_{\chi \in \text{Irr}(W)} \frac{1}{s_{\chi}} \chi_q$$

where $s_{\chi}$ is the Schur element of $\mathcal{H}_q(W)$ associated with $\chi$. 
We have that $s_\chi$ belongs to $A = \mathbb{Z}_K[q, q^{-1}]$ and it is a product of cyclotomic polynomials over $K$.

**Definition**

We define $a_\chi$ to be the smallest non-negative integer such that

$$q^{a_\chi} s_\chi \in \mathbb{Z}_K[q].$$

**Example:** If $s_\chi = q^{-1} + 2 + q$, then $a_\chi = 1$. 
The decomposition matrix

Let
\[ \theta : A \to \mathbb{C}, \quad q \mapsto \xi \]
be a ring homomorphism. Set \( \mathcal{H}_\xi := \mathbb{C} \otimes_A \mathcal{H}_q(W) \).

**Theorem (Geck-Pfeiffer)**

The algebra \( \mathcal{H}_\xi \) is semisimple if and only if \( \theta(s_\chi) \neq 0 \) for all \( \chi \in \text{Irr}(W) \).

We have a well-defined decomposition map

\[ d_\theta : R_0(K(q)\mathcal{H}_q(W)) \to R_0(\mathcal{H}_\xi) \]

with corresponding decomposition matrix

\[ D_\theta = \left( [E : M] \right)_{E \in \text{Irr}(W), M \in \text{Irr}(\mathcal{H}_\xi)} \]
We say that $\mathcal{H}_q(W)$ admits a canonical basic set $B^\text{can} \subset \text{Irr}(W)$ with respect to $\theta : A \to \mathbb{C}$ if there exists a bijection

$$\text{Irr}(\mathcal{H}_\xi) \leftrightarrow B^\text{can}$$

$$M \leftrightarrow E_M$$

such that

1. $[E_M : M] = 1$, and
2. if $[E : M] \neq 0$, then either $E = E_M$ or $a_E > a_{E_M}$. 
If $\mathcal{H}_q(W)$ admits a canonical basic set $B^{\text{can}}$ with respect to $\theta$, then the decomposition matrix $D_\theta$ has the following form:

$$
D_\theta = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
* & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & 1
\end{pmatrix}
$$

$$
\begin{cases}
B^{\text{can}} \\
\text{Irr}(\mathcal{H}_\xi) \\
\text{Irr}(W)
\end{cases}
$$
The algebra $\mathcal{H}_q(W)$ admits a canonical basic set with respect to any specialization $\theta : A \rightarrow \mathbb{C}$, if

1. $W$ is a Weyl group;  
   \cite{Geck-Rouquier, Geck, Geck-Jacon, C.-Jacon}

2. $W$ is a complex reflection group of type $G(d, 1, r)$;  
   \cite{Dipper-James-Murphy, Geck-Rouquier, Ariki, Uglov, Jacon}

3. $W$ is a complex reflection group of type $G(de, e, r)$  
   (for a certain choice of parameters);  
   \cite{Genet-Jacon}

4. $W \in \{G_4, G_5, G_8, G_9, G_{10}, G_{12}, G_{16}, G_{20}, G_{22}\}$  
   (for certain choices of parameters).  
   \cite{C.-Miyachi}
In the last case, we have been also able to show that there exists a subset $\mathcal{B}^{\text{opt}} \subset \text{Irr}(W)$ such that the decomposition matrix $D_\theta$ has the following form:

$$D_\theta = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix} \quad \left\{ \begin{array}{c}
\mathcal{B}^{\text{opt}} \\
\text{Irr}(W)
\end{array} \right\} \left\{ \begin{array}{c}
\text{Irr}(H_\xi) \\
\text{Irr}(W)
\end{array} \right\}$$