Schur elements and Rouquier blocks

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$$\begin{aligned} \mathcal{H}(G_4) = < S, T \,|\quad STS = TST, \quad & (S-u_0)(S-u_1)(S-u_2) = 0, \\ & (T-u_0)(T-u_1)(T-u_2) = 0 > . \end{aligned}$$

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- $u_j \mapsto \zeta_3^j$ $(j = 0,1,2)$, $\mathcal{H}(G_4) \mapsto \mathbb{Z}_K[G_4]$.

Assumptions

- The algebra $\mathcal{H}(W)$ is a free $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-module of rank $|W|$.
- There exists a unique linear form $t : \mathcal{H}(W) \to \mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ such that
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We have that

$$t = \sum_{\chi_{\mathbf{v}} \in \mathrm{Irr}(K_W(\mathbf{v})\mathcal{H}(W))} \frac{1}{s_{\chi_{\mathbf{v}}}} \chi_{\mathbf{v}},$$

where $s_{\chi_{\mathbf{v}}}$ is the Schur element associated with the irreducible character $\chi_{\mathbf{v}}$.

Cyclotomic Hecke algebras

Definition

Let y be an indeterminate. A cyclotomic specialization of $\mathcal{H}$ is a $\mathbb{Z}_K$-algebra morphism $\phi : v_j \mapsto y^{n_j}$ where $n_j \in \mathbb{Z}$ for all j. The corresponding cyclotomic Hecke algebra $\mathcal{H}_\phi$ is the $\mathbb{Z}_K[y, y^{-1}]$-algebra obtained as the specialization of the $\mathcal{H}$ via the morphism ϕ.

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The algebra $K_W(y)\mathcal{H}_\phi$ is split semisimple.

By "Tits' deformation theorem", we obtain that the specialization $v_j \mapsto 1$ induces the following bijections :

$$\begin{array}{ccccc} \mathrm{Irr}(K_W(\mathbf{v})\mathcal{H}) & \leftrightarrow & \mathrm{Irr}(K_W(y)\mathcal{H}_\phi) & \leftrightarrow & \mathrm{Irr}(W) \\ \chi_{\mathbf{v}} & \mapsto & \chi_\phi & \mapsto & \chi \end{array}$$

Rouquier blocks

Definition

We call Rouquier ring of K_W and denote by $\mathcal{R}_K(y)$ the $\mathbb{Z}_K$-subalgebra of $K_W(y)$

$$\mathcal{R}_K(y) := \mathbb{Z}_K[y, y^{-1}, (y^n - 1)^{-1}_{n \geq 1}].$$

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$$\text{For all } B \in \mathcal{BR}(\mathcal{H}_\phi) \text{ and } h \in \mathcal{H}_\phi, \sum_{\chi \in B} \frac{\chi_\phi(h)}{s_{\chi_\phi}} \in \mathcal{R}_K(y).$$

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W c.r.g.(non-Weyl) : Rouquier blocks $\equiv$?

Generic Schur elements

Theorem (C.)

The Schur element $s_\chi(\mathbf{v})$ associated with the character $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}$ is an element of $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ of the form

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

where

- ξ_χ is an element of $\mathbb{Z}_K$,
- N_χ is a degree zero monomial in $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$,
- I_χ is an index set,
- $(\Psi_{\chi,i})_{i\in I_\chi}$ is a family of K-cyclotomic polynomials in one variable,
- $(M_{\chi,i})_{i\in I_\chi}$ is a family of degree zero primitive monomials in $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$,
- $(n_{\chi,i})_{i\in I_\chi}$ is a family of positive integers.

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- If M is a $\mathfrak{p}$-essential monomial for W, then the hyperplane H_M defined by $\mathrm{Log}(M) = 0$, *i.e.*, $\sum_j a_j V_j = 0$, is called a $\mathfrak{p}$-essential hyperplane for W.

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Let $\phi : v_j \mapsto y^{n_j}$ be a cyclotomic specialization and $M = \prod_j v_j^{a_j}$ be a $\mathfrak{p}$-essential monomial for W. We have $\phi(M) = 1$ if and only if the powers n_j belong to the $\mathfrak{p}$-essential hyperplane H_M.

$\mathfrak{p}$-blocks and $\mathfrak{p}$-essential hyperplanes

Let $\mathcal{B}_{\mathfrak{p}}(\mathcal{H})$ be the partition of $\mathrm{Irr}(W)$ into blocks of $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]_{\mathfrak{p}}\mathcal{H}$.

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Main Theorem (C.)

For every $\mathfrak{p}$-essential hyperplane H for W, there exists a partition $\mathcal{B}_\mathfrak{p}^H(\mathcal{H})$ of $\mathrm{Irr}(W)$ with the following properties:

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3. The Rouquier blocks of $\mathcal{H}_\phi$ coincide with the partition generated by the partitions $\mathcal{B}_{\mathfrak{p}}(\mathcal{H}_\phi)$, where $\mathfrak{p}$ runs over the set of all prime ideals of $\mathbb{Z}_K$.

The characters of G_4 are denoted by $\chi_{d,b}$, where d is their degree and b is the valuation of their fake degree. These are $\chi_{1,0}, \chi_{1,4}, \chi_{1,8}, \chi_{2,5}, \chi_{2,3}, \chi_{2,1}, \chi_{3,2}$.

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2-essential in purple, 3-essential in green

$$s_{1,0} = \Phi''_9(v_0v_1^{-1}) \cdot \Phi'_{18}(v_0v_1^{-1}) \cdot \Phi_4(v_0v_1^{-1}) \cdot \Phi'_{12}(v_0v_1^{-1}) \cdot \Phi''_{12}(v_0v_1^{-1}) \cdot \Phi'_{36}(v_0v_1^{-1}) \cdot \Phi'_9(v_0v_2^{-1}) \cdot \Phi''_{18}(v_0v_2^{-1}) \cdot \Phi_4(v_0v_2^{-1}) \cdot \Phi'_{12}(v_0v_2^{-1}) \cdot \Phi''_{12}(v_0v_2^{-1}) \cdot \Phi''_{36}(v_0v_2^{-1}) \cdot \Phi_4(v_0^2v_1^{-1}v_2^{-1}) \cdot \Phi'_{12}(v_0^2v_1^{-1}v_2^{-1}) \cdot \Phi''_{12}(v_0^2v_1^{-1}v_2^{-1})$$

$$s_{2,5} = -\zeta_3^2v_2^6v_1^{-6}\Phi'_9(v_1v_0^{-1}) \cdot \Phi''_{18}(v_1v_0^{-1}) \cdot \Phi''_9(v_2v_0^{-1}) \cdot \Phi'_{18}(v_2v_0^{-1}) \cdot \Phi_4(v_1v_2^{-1}) \cdot \Phi'_{12}(v_1v_2^{-1}) \cdot \Phi''_{12}(v_1v_2^{-1}) \cdot \Phi'_{36}(v_1v_2^{-1}) \cdot \Phi_4(v_0^{-2}v_1v_2) \cdot \Phi'_{12}(v_0^{-2}v_1v_2) \cdot \Phi''_{12}(v_0^{-2}v_1v_2)$$

$$s_{3,2} = \Phi_4(v_0^2v_1^{-1}v_2^{-1}) \cdot \Phi'_{12}(v_0^2v_1^{-1}v_2^{-1}) \cdot \Phi''_{12}(v_0^2v_1^{-1}v_2^{-1}) \cdot \Phi_4(v_1^2v_2^{-1}v_0^{-1}) \cdot \Phi'_{12}(v_1^2v_2^{-1}v_0^{-1}) \cdot \Phi''_{12}(v_1^2v_2^{-1}v_0^{-1}) \cdot \Phi_4(v_2^2v_0^{-1}v_1^{-1}) \cdot \Phi'_{12}(v_2^2v_0^{-1}v_1^{-1}) \cdot \Phi''_{12}(v_2^2v_0^{-1}v_1^{-1})$$

$\Phi_4(x) = x^2+1$, $\Phi'_9(x) = x^3-\zeta_3$, $\Phi''_9(x) = x^3-\zeta_3^2$, $\Phi''_{12}(x) = x^2+\zeta_3$, $\Phi'_{12}(x) = x^2+\zeta_3^2$, $\Phi''_{18}(x) = x^3+\zeta_3$, $\Phi'_{18}(x) = x^3+\zeta_3^2$, $\Phi''_{36}(x) = x^6+\zeta_3$, $\Phi'_{36}(x) = x^6+\zeta_3^2$.

The essential monomials for G_4 are

$$M_{0,1} := v_0 v_1^{-1},\ M_{0,2} := v_0 v_2^{-1},\ M_{1,2} := v_1 v_2^{-1},$$

$$M_0 := v_0^2 v_1^{-1} v_2^{-1},\ M_1 := v_1^2 v_2^{-1} v_0^{-1},\ M_2 := v_2^2 v_0^{-1} v_1^{-1}.$$

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Let c_0, c_1, c_2 be three indeterminates. The corresponding essential hyperplanes for G_4 in $\mathbb{C}^3$ are given by

$$H_{0,1} : c_0 = c_1,\ H_{0,2} : c_0 = c_2,\ H_{1,2} : c_1 = c_2,$$

$$H_0 : 2c_0 = c_1 + c_2,\ H_1 : 2c_1 = c_2 + c_0,\ H_2 : 2c_2 = c_0 + c_1.$$

They are all 2-essential, whereas only the first three are 3-essential.

The essential monomials for G_4 are

$$M_{0,1} := v_0 v_1^{-1},\ M_{0,2} := v_0 v_2^{-1},\ M_{1,2} := v_1 v_2^{-1},$$

$$M_0 := v_0^2 v_1^{-1} v_2^{-1},\ M_1 := v_1^2 v_2^{-1} v_0^{-1},\ M_2 := v_2^2 v_0^{-1} v_1^{-1}.$$

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We have created the GAP function

$$EssentialHyperplanes(W, p)$$

which returns the above information for any exceptional irreducible complex reflection group W.

Hyperplane	$\mathcal{B}_2^H(\mathcal{H}) \cup \mathcal{B}_3^H(\mathcal{H})$	$\mathcal{B}_2^H(\mathcal{H})$	$\mathcal{B}_3^H(\mathcal{H})$
None	-	-	-
$H_{0,1}$	$(\chi_{1,0}, \chi_{1,4}, \chi_{2,1})$, $(\chi_{2,5}, \chi_{2,3})$	$(\chi_{1,0}, \chi_{1,4}, \chi_{2,1})$	$(\chi_{1,0}, \chi_{1,4})$, $(\chi_{2,5}, \chi_{2,3})$
$H_{0,2}$	$(\chi_{1,0}, \chi_{1,8}, \chi_{2,3})$, $(\chi_{2,5}, \chi_{2,1})$	$(\chi_{1,0}, \chi_{1,8}, \chi_{2,3})$	$(\chi_{1,0}, \chi_{1,8})$, $(\chi_{2,5}, \chi_{2,1})$
$H_{1,2}$	$(\chi_{1,4}, \chi_{1,8}, \chi_{2,5})$, $(\chi_{2,3}, \chi_{2,1})$	$(\chi_{1,4}, \chi_{1,8}, \chi_{2,5})$	$(\chi_{1,4}, \chi_{1,8})$, $(\chi_{2,3}, \chi_{2,1})$
H_0	$(\chi_{1,0}, \chi_{2,5}, \chi_{3,2})$	$(\chi_{1,0}, \chi_{2,5}, \chi_{3,2})$	-
H_1	$(\chi_{1,4}, \chi_{2,3}, \chi_{3,2})$	$(\chi_{1,4}, \chi_{2,3}, \chi_{3,2})$	-
H_2	$(\chi_{1,8}, \chi_{2,1}, \chi_{3,2})$	$(\chi_{1,8}, \chi_{2,1}, \chi_{3,2})$	-

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$H_{0,2}$	$(\chi_{1,0}, \chi_{1,8}, \chi_{2,3})$, $(\chi_{2,5}, \chi_{2,1})$	$(\chi_{1,0}, \chi_{1,8}, \chi_{2,3})$	$(\chi_{1,0}, \chi_{1,8})$, $(\chi_{2,5}, \chi_{2,1})$
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H_1	$(\chi_{1,4}, \chi_{2,3}, \chi_{3,2})$	$(\chi_{1,4}, \chi_{2,3}, \chi_{3,2})$	-
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We have created the GAP functions

$$\mathit{DisplayAllBlocks}(W) \text{ and } \mathit{DisplayAllpBlocks}(W)$$

which display the above information for any exceptional irreducible complex reflection group W.

Calculation of the Rouquier blocks of a cyclotomic Hecke algebra

Calculation of the Rouquier blocks of a cyclotomic Hecke algebra

$$\mathcal{H}_\phi \;=\; < S, T \mid \; STS = TST, \;\; \begin{array}{l}(S-1)(S-\zeta_3 x)(S-\zeta_3^2 x^2) = 0 \\ (T-1)(T-\zeta_3 x)(T-\zeta_3^2 x^2) = 0 >\end{array}$$

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For any exceptional irreducible complex reflection group W, the GAP function

$$\mathit{DisplayRouquierBlocks}(H)$$

displays the Rouquier blocks of a given cyclotomic Hecke algebra H.

Calculation of the Rouquier blocks of the group algebra

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Rouquier block	$(\chi_{1,0}, \chi_{1,4}, \chi_{1,8}, \chi_{2,5}, \chi_{2,3}, \chi_{2,1}, \chi_{3,2})$
2-block	$(\chi_{1,0}, \chi_{1,4}, \chi_{1,8}, \chi_{2,5}, \chi_{2,3}, \chi_{2,1}, \chi_{3,2})$

Calculation of the Rouquier blocks of the group algebra

$$\mathbb{Z}[\zeta_3][G_4] \quad \simeq \quad < S, T \mid \quad STS = TST, \quad \begin{array}{l}(S-1)(S-\zeta_3)(S-\zeta_3^2) = 0 \\ (T-1)(T-\zeta_3)(T-\zeta_3^2) = 0 >\end{array}$$

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$$\begin{array}{rl} \text{Rouquier block} & (\chi_{1,0}, \chi_{1,4}, \chi_{1,8}, \chi_{2,5}, \chi_{2,3}, \chi_{2,1}, \chi_{3,2}) \\ \text{2-block} & (\chi_{1,0}, \chi_{1,4}, \chi_{1,8}, \chi_{2,5}, \chi_{2,3}, \chi_{2,1}, \chi_{3,2}) \\ \text{3-blocks} & (\chi_{1,0}, \chi_{1,4}, \chi_{1,8}), (\chi_{2,5}, \chi_{2,3}, \chi_{2,1}), (\chi_{3,2}) \end{array}$$

Let y be an indeterminate and $n \in \mathbb{Z}$. Then

$$(y^n)^+ := \begin{cases} n, & \text{if } n > 0; \\ 0, & \text{if } n \leq 0. \end{cases} \text{ and } (y^n)^- := \begin{cases} n, & \text{if } n < 0; \\ 0, & \text{if } n \geq 0. \end{cases}$$

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Let $\chi \in \mathrm{Irr}(W)$. The generic Schur element $s_\chi(\mathbf{v})$ is of the form

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}} \qquad (\dagger).$$

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Theorem (C.)

Let W be an exceptional complex reflection group and $\chi, \psi \in \mathrm{Irr}(W)$. If the characters χ_ϕ and ψ_ϕ belong to the same Rouquier block, then

$$a_{\chi_\phi} = a_{\psi_\phi} \text{ and } A_{\chi_\phi} = A_{\psi_\phi}.$$