Basic sets for cyclotomic Hecke algebras

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Iwahori-Hecke algebras

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- q be an indeterminate.

The Iwahori-Hecke algebra $\mathcal{H}$ of W is generated over $A := \mathbb{Z}[q, q^{-1}]$ by the elements $(T_w)_{w \in W}$, whose multiplication is determined by the following rules:

$$\begin{cases} T_w T_{w'} = T_{ww'}, \text{ if } \ell(ww') = \ell(w) + \ell(w'). \\ (T_s - q^{2u_s})(T_s + q^{2v_s}) = 0, \text{ for all } s \in S. \end{cases}$$

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then

$$\mathrm{Irr}(K\mathcal{H}) = \{V^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Lambda\}.$$

The a-function

The linear form $t : \mathcal{H} \to A$ given by

$$t(T_w) = \begin{cases} 1, & \text{if } w = 1 \\ 0, & \text{if } w \neq 1 \end{cases}$$

is a symmetrizing form for the Iwahori-Hecke algebra $\mathcal{H}$,

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1. $t(hh') = t(h'h)$,
2. the bilinear form $(h, h') \mapsto t(hh')$ is non-degenerate.

We have

$$t = \sum_{V \in \mathrm{Irr}(K\mathcal{H})} \frac{1}{s_V} \chi_V$$

where $s_V \in A$ is the Schur element of the irreducible character χ_V.

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Definition

Recall that $\mathrm{Irr}(K\mathcal{H}) = \{V^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Lambda\}$. Set $s_{\boldsymbol{\lambda}} := s_{V^{\boldsymbol{\lambda}}}$ and

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Example: If $s_{\boldsymbol{\lambda}} = q^{-2} + q^{-1} + q^3$, then $a_{\boldsymbol{\lambda}} = 2$.

The decomposition matrix

Let

$$\theta : A = \mathbb{Z}[q, q^{-1}] \to L, \ q \mapsto \xi$$

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Assume that $L\mathcal{H}$ is split. Let $R_0(K\mathcal{H})$ (resp. $R_0(L\mathcal{H})$) be the Grothendieck group of finitely generated $K\mathcal{H}$-modules (resp. $L\mathcal{H}$-modules). It is generated by the classes $[U]$ of the simple $K\mathcal{H}$-modules (resp. $L\mathcal{H}$-modules) U.

We have a well-defined decomposition map

$$d_\theta : R_0(K\mathcal{H}) \to R_0(L\mathcal{H})$$

such that, for all $\boldsymbol{\lambda} \in \Lambda$, we have

$$d_\theta([V^{\boldsymbol{\lambda}}]) = \sum_{M \in \mathrm{Irr}(L\mathcal{H})} [V^{\boldsymbol{\lambda}} : M][M].$$

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The matrix

$$D_\theta = \left([V^{\boldsymbol{\lambda}} : M]\right)_{\boldsymbol{\lambda} \in \Lambda,\, M \in \mathrm{Irr}(L\mathcal{H})}$$

is called the decomposition matrix associated with θ.

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2. The map

$$\begin{array}{ccc} \mathrm{Irr}(L\mathcal{H}) & \to & \mathcal{B} \\ M & \mapsto & \boldsymbol{\lambda}_M \end{array}$$

   is a bijection.

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Theorem (C.-Jacon)

Let L be a field of characteristic 0. The algebra $\mathcal{H}$ admits a canonical basic set with respect to any specialization $\theta : A \longrightarrow L$.

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$$(v_s - u_s, 0) \to (0, v_s - u_s) \to (u_s, v_s),\ \mathcal{B} \mapsto \mathcal{B}^{\varepsilon} = \{\varepsilon(\boldsymbol{\lambda}) \,|\, \boldsymbol{\lambda} \in \mathcal{B}\}$$

where $\varepsilon : \Lambda \to \Lambda$ is a bijection induced on the set of irreducible representations of $K\mathcal{H}$ by the action of the cyclic group $\mathbb{Z}/2\mathbb{Z}$.

An example: Type A

Let W be the symmetric group $\mathfrak{S}_n$ and S the set of transpositions $(i, i+1)$, for $i = 1, \ldots, n-1$.

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for some integers u, v.

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The indexing set Λ for $\mathrm{Irr}(W)$ is the set

$$\mathcal{P}(n) := \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_r) \,\middle|\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1 \text{ and } \sum_{i=1}^{r} \lambda_i = n \right\}$$

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of partitions of n. The bijection ε on Λ induced by the action of $\mathbb{Z}/2\mathbb{Z}$ is simply the conjugation of partitions.

Proposition (Dipper-James, C.-Jacon)

Let $\theta : \mathbb{Z}[q, q^{-1}] \to L$ be a specialization such that $\theta(q)^{2u-2v}$ is a primitive root of 1 of order $e > 1$.

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1. If $u > v$, then $\mathcal{H}$ admits a canonical basic set $\mathcal{B} \subset \Lambda$ with respect to θ and we have
$$\mathcal{B} = \mathrm{Reg}_e(n)$$
where the set $\mathrm{Reg}_e(n)$ of e-regular partitions is defined by
$$\lambda = (\lambda_1, \ldots, \lambda_r) \notin \mathrm{Reg}_e(n) \iff \exists i \in \mathbb{N},\ \lambda_i = \cdots = \lambda_{i+e-1} \neq 0.$$

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2. If $u < v$, then $\mathcal{H}$ admits a canonical basic set $\mathcal{B} \subset \Lambda$ with respect to θ and we have
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