THE YOKONUMA-HECKE ALGEBRA OF TYPE A

Maria Chlouveraki (UVSQ)

Perspectives in Lie Theory

Braid group (of type A)

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i = 1, \ldots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| > 1 \end{array} \right\rangle$$

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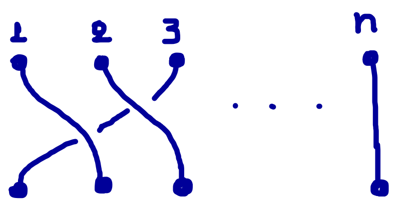

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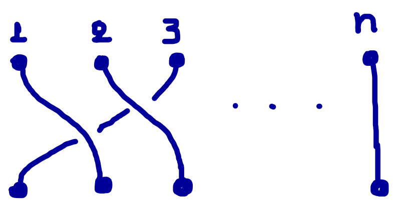
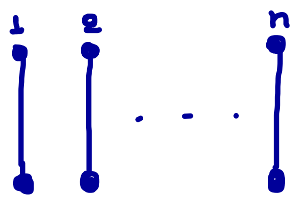
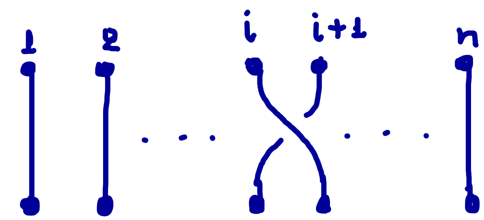
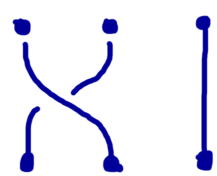
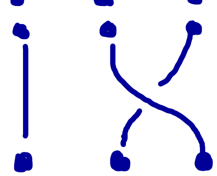


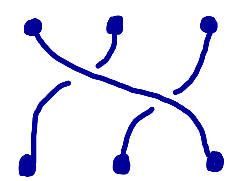
Identity = 

σ_i = 

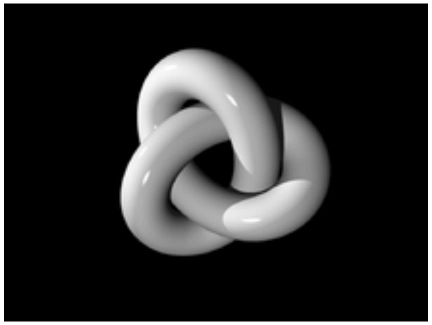
Multiplication : concatenation of diagrams

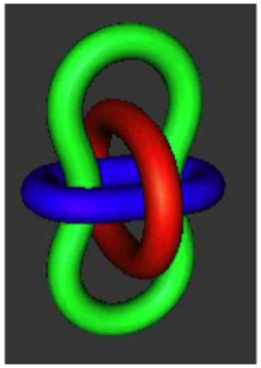
E.g. $\alpha = \sigma_1 =$ 

$\beta = \sigma_2 =$ 

$\Rightarrow \alpha\beta = \sigma_1\sigma_2 =$ 

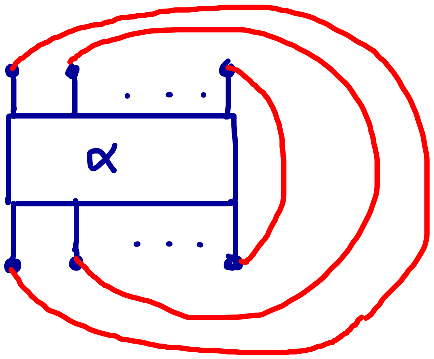
A **knot** (respectively a **link**) is an embedding of the circle S^1 (resp. n copies of S^1) into 3-dimensional Euclidean space $\mathbb{R}^3$.





Every link can be represented by a "knot diagram"

Every element of B_n gives rise to a knot or link



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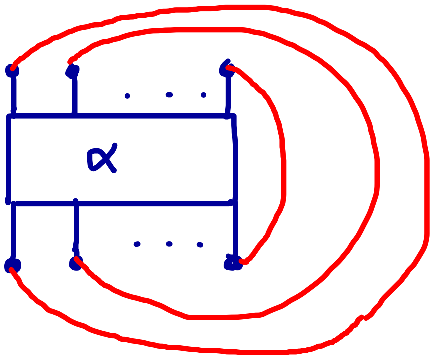

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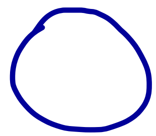
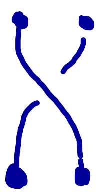
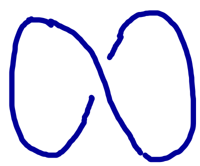
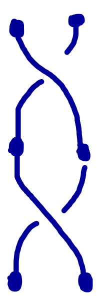
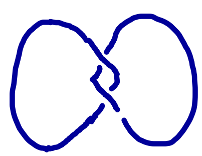
<u>E.g.</u> $\alpha =$  $\hat{\alpha} =$ 

$\alpha = \sigma_1 =$  $\hat{\alpha} =$ 

$\alpha = \sigma_1^2 =$  $\hat{\alpha} =$ 

$\alpha = \sigma_1^3$ $\hat{\alpha}$ = left trefoil knot

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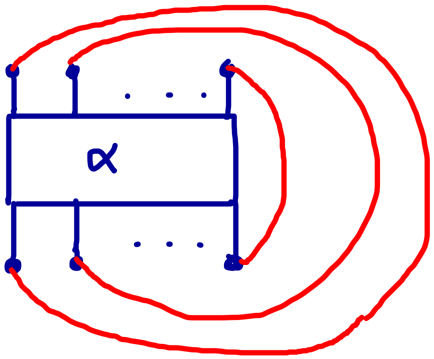

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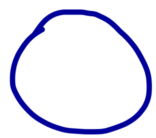
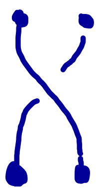
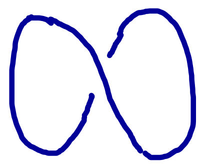
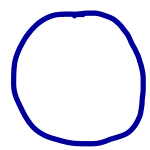
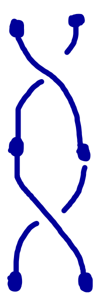
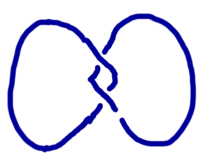
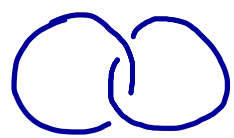
<u>E.g.</u> $\alpha =$  $\hat{\alpha} =$ 

$\alpha = \sigma_1 =$  $\hat{\alpha} =$  $\sim$ 

$\alpha = \sigma_1^2 =$  $\hat{\alpha} =$  $\sim$ 

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Alexander's Theorem (1923)

Every link can be obtained as the closure $\hat{\alpha}$ of a braid $\alpha \in \bigcup_{n \geq 1} B_n$.

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We define an equivalence relation on $\bigcup_{n \geq 1} B_n$ as the transitive closure of the relations:

(i) $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in B_n$ (conjugation)

(ii) $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n$ (Markov's move)

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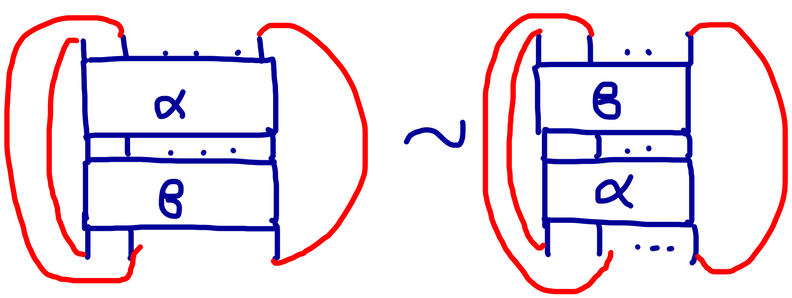
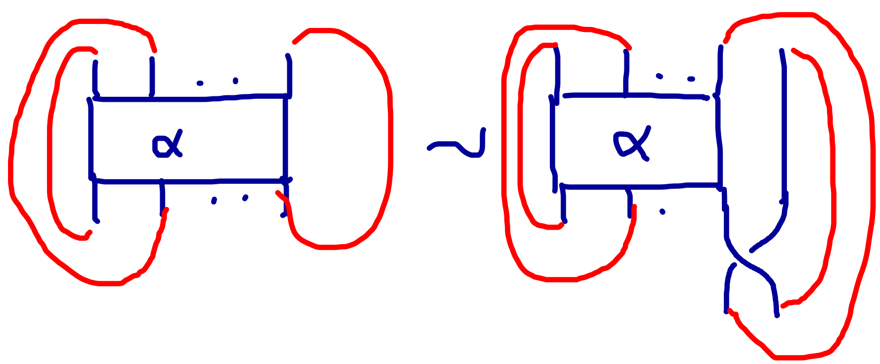
Markov's Theorem (1935)

We have $\hat{\alpha} \sim \hat{\beta}$ if and only if $\alpha \sim \beta$.





$\mathcal{L}$ = set of links

A knot invariant is a function $I : \mathcal{L} \longrightarrow S$ (S a set)

such that

$$L_1 \sim L_2 \implies I(L_1) = I(L_2)$$

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Equivalently, a knot invariant is a function $I : \bigcup_{n \geq 1} B_n \longrightarrow S$

such that

(i) $I(\alpha\beta) = I(\beta\alpha) \quad \forall\ \alpha, \beta \in B_n$

(ii) $I(\alpha) = I(\alpha\sigma_n) = I(\alpha\sigma_n^{-1}) \quad \forall\ \alpha \in B_n$

Here S will be a set of polynomials.

Iwahori - Hecke algebra of type A

q indeterminate, $R = \mathbb{C}[q, q^{-1}]$

$$\mathcal{H}_n(q) = \left\langle G_1, \dots, G_{n-1} \;\middle|\; \begin{array}{l} G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} \\ G_i G_j = G_j G_i \quad \text{if } |i-j| > 1 \\ G_i^2 = (q-1) G_i + q \end{array} \right\rangle$$

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Normalisation of τ $\longrightarrow$ HOMFLYPT (or 2-variable Jones) polynomial

Framed braid group

Let $d \in \mathbb{Z}_{>0}$

$$(\mathbb{Z}/d\mathbb{Z}) \wr B_n = \left\langle \sigma_1, \dots, \sigma_{n-1}, t_1, \dots, t_n \;\middle|\; \begin{array}{l} \sigma_i \text{ as before} \\ t_j^d = 1 \quad j = 1, \dots, n \\ t_i t_j = t_j t_i \quad i, j = 1, \dots, n \\ t_j \sigma_i = \sigma_i t_{s_i(j)} \end{array} \right\rangle$$

$$\rotatebox{90}{=}$$

$$(\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$$

where $s_i = (i, i+1) \in S_n$

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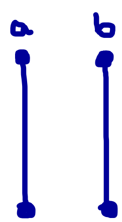
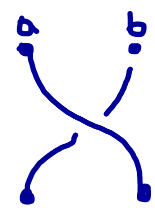
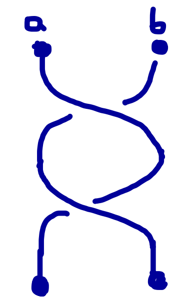
$t_1^a t_2^b =$ 

$t_1^a t_2^b \sigma_1 =$ 

$t_1^a t_2^b \sigma_1^2 =$ 

Multiplication : concatenation of diagrams

$$(t_1^a t_2^b \sigma_1)(t_1^{a'} t_2^{b'}) = t_1^{a+b'} t_2^{b+a'} \sigma_1$$

Every element of $(\mathbb{Z}/d\mathbb{Z}) \wr B_n$ gives rise to a framed knot or link

<u>E.g.</u> $a, b \in \{0, 1, \dots, d-1\}$, $n = 2$

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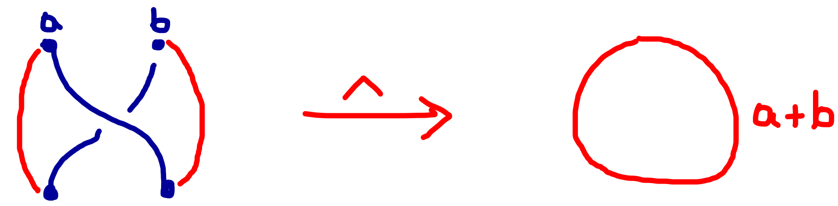

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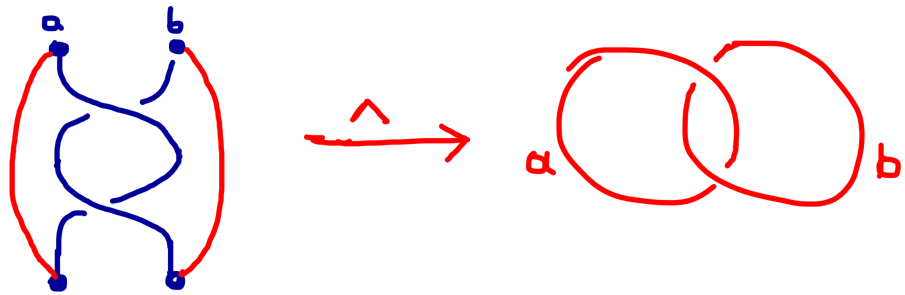

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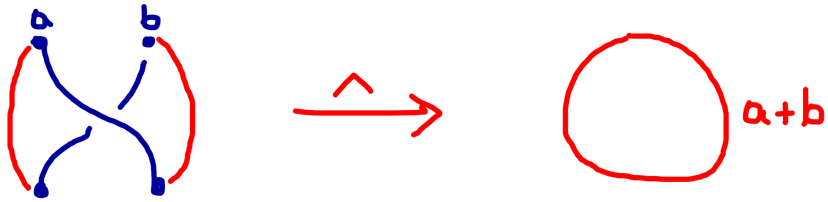

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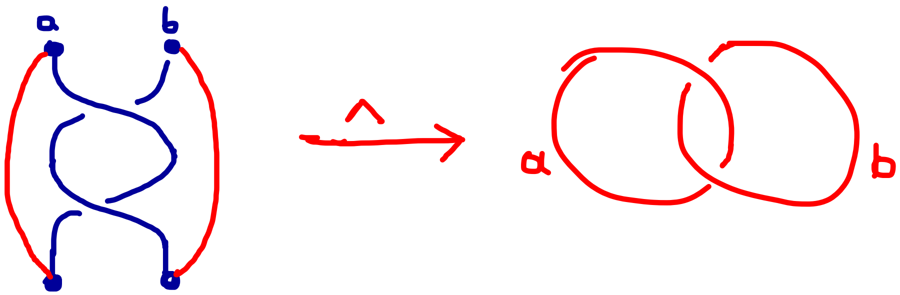

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Alexander's Theorem: obvious

Markov's Theorem: Ko-Smolinsky 1992

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where $e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}$ is an idempotent.

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- $q = 1$: $Y_{d,n}(1) \cong \mathbb{C}[G(d,1,n)]$, where $G(d,1,n) = (\mathbb{Z}/d\mathbb{Z}) \wr S_n$

Yokonuma-Hecke algebra of type A

q indeterminate, $R = \mathbb{C}[q, q^{-1}]$

$$Y_{d,n}(q) = \left\langle \begin{array}{c} g_1, \dots, g_{n-1} \\ t_1, \dots, t_n \end{array} \middle| \begin{array}{l} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \quad \text{if} \quad |i-j| > 1 \\ t_j^d = 1 \quad , \quad t_i t_j = t_j t_i \quad , \quad t_j g_i = g_i t_{s_i(j)} \\ \qquad g_i^2 = (q-1) e_i g_i + q \end{array} \right\rangle$$

where $e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}$ is an idempotent.

- $Y_{d,n}(q)$ is a quotient of $R[(\mathbb{Z}/d\mathbb{Z}) \wr B_n]$
- $q = 1$: $Y_{d,n}(1) \cong \mathbb{C}[G(d,1,n)]$, where $G(d,1,n) = (\mathbb{Z}/d\mathbb{Z}) \wr S_n$
- $d = 1$: $Y_{1,n}(q) \cong \mathcal{H}_n(q)$

Yokonuma - Hecke algebra of type A

q indeterminate, $R = \mathbb{C}[q, q^{-1}]$

$$Y_{d,n}(q) = \left\langle \begin{matrix} g_1, \dots, g_{n-1} \\ t_1, \dots, t_n \end{matrix} \;\middle|\; \begin{matrix} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \quad \text{if } |i-j| > 1 \\ t_j^d = 1 \quad, \quad t_i t_j = t_j t_i \quad, \quad t_j g_i = g_i t_{s_i(j)} \\ g_i^2 = (q-1) e_i g_i + q \end{matrix} \right\rangle$$

where $e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}$ is an idempotent.

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- $d = 1$: $Y_{1,n}(q) \cong \mathcal{H}_n(q)$
- $\mathcal{H}_n(q)$ is a quotient of $Y_{d,n}(q)$ $\quad (t_j \longmapsto 1)$

There exists a Markov trace $tr : Y_{d,n}(q) \longrightarrow R$, the Juyumaya trace, depending on parameters $z, x_0, x_1, \ldots, x_{d-1}$.

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Normalisation of tr

$\downarrow$ E-condition on $(x_0, x_1, \ldots, x_{d-1})$

Juyumaya-Lambropoulou invariant for framed knots and links

There exists a Markov trace $tr : Y_{d,n}(q) \longrightarrow R$, the Juyumaya trace, depending on parameters $z, x_0, x_1, \dots, x_{d-1}$.

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Normalisation of tr

$\Big\downarrow$ E-condition on $(x_0, x_1, \dots, x_{d-1})$

Juyumaya-Lambropoulou invariant for framed knots and links

$\Big\downarrow$ "Forget" the framings

Invariant for classical knots and links

There exists a Markov trace $\mathrm{tr}: Y_{d,n}(q) \longrightarrow R$, the Juyumaya trace, depending on parameters $z, x_0, x_1, \dots, x_{d-1}$.

$$\boxed{(\mathbb{Z}/d\mathbb{Z}) \wr B_n \hookrightarrow R[(\mathbb{Z}/d\mathbb{Z}) \wr B_n] \twoheadrightarrow Y_{d,n}(q) \xrightarrow{\mathrm{tr}} R}$$

$$\begin{array}{c} \text{Normalisation of tr} \\ \Big\downarrow \; \text{E-condition on } (x_0, x_1, \dots, x_{d-1}) \\ \text{Juyumaya-Lambropoulou} \\ \text{invariant for framed knots and links} \\ \Big\downarrow \; \text{"Forget" the framings} \\ \text{Invariant for classical knots and links} \end{array}$$

Invariant for classical knots and links = HOMFLYPT

- when $\mathrm{tr}(e_i) = 1$ [CL]
- on knots [CCJKL]

Representation theory of $Y_{d,n}(q)$

$$\boxed{\mathrm{Irr}(\mathbb{C}(q)Y_{d,n}(q)) \longleftrightarrow \mathrm{Irr}(G(d,1,n)) \longleftrightarrow \{\text{d-partitions of } n\}}$$

A d-partition of n is a family of d partitions $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ such that $|\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(d)}| = n$.

- Thiem 2005 : Unipotent Hecke algebras

- C.-Poulain d'Andecy : Jucys-Murphy elements
  Explicit combinatorial formulas
  Semisimplicity criterion / Schur elements

$\mathbb{C}[G(d,1,n)]$

Yokonuma-Hecke algebra

- Wreath product
- Braid group of type A
- Rep^n theory of $\mathcal{H}_n(q)$

Ariki-Koike algebra

- Quadratic relation for g_i
- Braid group of type B
- $\mathcal{H}_n(q)$ (obvious) subalgebra

Cyclotomic Yokonuma-Hecke algebra $Y(d,m,n)$

[C. - Poulain d'Andecy]

Yokonuma-Hecke algebra

$Y_{d,n}(q) = Y(d,1,n)$

Ariki-Koike algebra

$Y(1,m,n)$

Cyclotomic Yokonuma-Hecke algebra $Y(d,m,n)$

[C. - Poulain d'Andecy]

Yokonuma-Hecke algebra	Ariki-Koike algebra
$Y_{d,n}(q) = Y(d,1,n)$	$Y(1,m,n)$

- Markov trace on the Ariki-Koike algebra [Lambropoulou, Geck-Lambropoulou 1994-1999]

$\downarrow$

Invariant for knots in the solid torus

Cyclotomic Yokonuma-Hecke algebra $Y(d,m,n)$

[C. - Poulain d'Andecy]

Yokonuma-Hecke algebra

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- Markov trace on the Ariki-Koike algebra [Lambropoulou, Geck-Lambropoulou 1994-1999]

$\downarrow$

Invariant for knots in the solid torus

- Markov trace on $Y(d,m,n)$

$\downarrow$ E-condition

Invariant for framed knots in the solid torus

$\downarrow$ Forget the framings

Invariant for knots in the solid torus

Temperley-Lieb algebra

Let $n \geqslant 3$.

$$TL_n(q) := \mathcal{H}_n(q) \Big/ \langle 1 + G_1 + G_2 + G_1G_2 + G_2G_1 + G_1G_2G_1 \rangle$$

Temperley-Lieb algebra

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$$\mathrm{Irr}\,(\mathbb{C}(q)\mathcal{H}_n(q)) \longleftrightarrow \{\text{partitions of } n\}$$

$$\mathrm{Irr}\,(\mathbb{C}(q)\,TL_n(q)) \longleftrightarrow \left\{\begin{array}{c}\text{partitions of } n \text{ whose Young diagrams} \\ \text{have at most 2 columns}\end{array}\right\}$$

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$$\mathrm{Irr}\left(\mathbb{C}(q)\,TL_n(q)\right) \longleftrightarrow \left\{\begin{array}{c}\text{partitions of } n \text{ whose Young diagrams} \\ \text{have at most 2 columns}\end{array}\right\}$$

Basis $\nearrow$ Easily extracted from the standard basis of $\mathcal{H}_n(q)$

Basis $\searrow$ Diagrammatic

Yokonuma-Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$YTL_{d,n}(q) := Y_{d,n}(q) \big/ \langle 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1 \rangle$$

Yokonuma-Temperley-Lieb algebra

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[C.-Pouchin]

$$\mathrm{Irr}\left(\mathbb{C}(q) YTL_{d,n}(q)\right) \longleftrightarrow \left\{ \begin{array}{l} d\text{-partitions } \lambda = (\lambda^{(1)}, \dots, \lambda^{(d)}) \text{ of } n \\ \text{such that the Young diagrams of} \\ \text{all } \lambda^{(i)} \text{ together have at most 2 columns} \end{array} \right\}$$

Yokonuma-Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

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Basis : extracted from the standard basis of $Y_{d,n}(q)$ (with difficulty!)

Framisation of the Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$FTL_{d,n}(q) := Y_{d,n}(q) / \langle e_1 e_2 (1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1) \rangle$$

Framisation of the Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$FTL_{d,n}(q) := Y_{d,n}(q) \big/ \langle e_1 e_2 \,(1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1) \rangle$$

[C.-Pouchin]

$$\mathrm{Irr}\,(\mathbb{C}(q)\,FTL_{d,n}(q)) \longleftrightarrow \left\{ \begin{array}{l} d\text{-partitions } \lambda = (\lambda^{(1)}, \dots, \lambda^{(d)}) \text{ of } n \\ \text{such that the Young diagram of} \\ \text{each } \lambda^{(i)} \text{ has at most 2 columns} \end{array} \right\}$$

Framisation of the Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$FTL_{d,n}(q) := Y_{d,n}(q) \,/ \left\langle e_1 e_2 \left(1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1 \right) \right\rangle$$

[C.-Pouchin]

$$\mathrm{Irr}\left(\mathbb{C}(q)\, FTL_{d,n}(q)\right) \longleftrightarrow \left\{ \begin{array}{l} d\text{-partitions } \lambda = (\lambda^{(1)}, \dots, \lambda^{(d)}) \text{ of } n \\ \text{such that the Young diagram of} \\ \text{each } \lambda^{(i)} \text{ has at most 2 columns} \end{array} \right\}$$

Basis : Work in progress!

The affine Yokonuma-Hecke algebra $Y(d,\infty,n)$

[C.-Poulain d'Andecy]

$$\begin{tikzcd} \text{Iwahori-Hecke algebra} \arrow[r, "\text{Replace Borel}", "\text{by max}^{l}\text{ unipotent}"'] \arrow[d, "\text{affinization}"] & \text{Yokonuma-Hecke algebra} \arrow[d, "\text{affinization}"] \\ \text{Affine Hecke algebra} & \text{Affine Y-H algebra } Y(d,\infty,n) \end{tikzcd}$$

The affine Yokonuma-Hecke algebra $Y(d,\infty,n)$

[C.-Poulain d' Andecy]

$$\begin{tikzcd}[column sep=huge, row sep=large]
\text{Iwahori-Hecke algebra} \arrow[r, "\text{Replace Borel}", "\text{by max}^{\text{l}}\text{ unipotent}"'] \arrow[d, "\text{affinization}"] & \text{Yokonuma-Hecke algebra} \arrow[d, "\text{affinization}"] \\
\text{Affine Hecke algebra} \arrow[r, "\text{Replace Iwahori sbgp}", "\text{by pro-p-Iwahori sbgp}"'] & \text{Affine Y-H algebra } Y(d,\infty,n) \arrow[d, phantom, "\overset{?}{\wr}\ \text{[C.- Sécherre]}"] \\
& \text{pro-p-Iwahori-Hecke algebra}
\end{tikzcd}$$

pro-p-Iwahori-Hecke algebra

[Vignéras]