Families of characters of the imprimitive complex reflection groups

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Expansion of Combinatorial Representation Theory

RIMS Workshop 2008
A finite reflection group on $K$ is a finite subgroup of $\text{GL}_K(V)$ ($V$ a finite-dimensional $K$-vector space) generated by pseudo-reflections, i.e., linear maps whose vector space of fixed points is a hyperplane.

A finite reflection group on $Q$ is called a Weyl group.

A finite reflection group on $R$ is called a (finite) Coxeter group.

A finite reflection group on $C$ is called a complex reflection group.
Complex reflection groups

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- either there exist positive integers $d, e, r$ such that $W$ is isomorphic to $G(de, e, r)$, where $G(de, e, r)$ is the group of all $r \times r$ monomial matrices with non-zero entries in $\mu_{de}$ and product of the non-zero entries in $\mu_d$,
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- or $W$ is isomorphic to an exceptional group $G_n$ ($n = 4, \ldots, 37$).
Every complex reflection group $W$ has a Coxeter-like presentation:

\[ G_2 = \langle s, t \mid ststst = tststs, s^2 = t^2 = 1 \rangle, \]

\[ G_4 = \langle s, t \mid sts = tst, s^3 = t^3 = 1 \rangle, \]

and a field of realization $K$:

\[ K_{G_2} = \mathbb{Q}, \]

\[ K_{G_4} = \mathbb{Q}(\zeta_3). \]

We choose a set of indeterminates $u = (u_s, j)$, $0 \leq j \leq o(s) - 1$, where $s$ runs over the set of generators of $W$ and $o(s)$ denotes the order of $s$ (if $s$ and $t$ are conjugate in $W$, then $u_s, j = u_t, j$ for all $j$).

The associated generic Hecke algebra $H(W)$ is an algebra over the Laurent polynomial ring $\mathbb{Z}[u, u^{-1}]$ and has a presentation of the form:

\[ H(G_2) = \langle S, T \mid STSTST = TSTSTS, (S - u_0)(S - u_1) = 0, (T - w_0)(T - w_1) = 0 \rangle, \]

\[ H(G_4) = \langle S, T \mid STS = TST, (S - u_0)(S - u_1)(S - u_2) = 0, (T - u_0)(T - u_1)(T - u_2) = 0 \rangle. \]
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Remark: The specialization $u_{s,j} \mapsto \zeta_{\omega(s)}^j$ sends $\mathcal{H}(W)$ to $\mathbb{Z}_K W$. 
Remark: The specialization $u_{s,j} \mapsto \zeta_{o(s)}^j$ sends $\mathcal{H}(W)$ to $\mathbb{Z}_K W$.

**Theorem (Malle)**

Let $\mathbf{v} = (v_{s,j})_{s,j}$ be a set of indeterminates such that, for all $s,j$, we have

$$v_{s,j} |_{\mu(K)} := \zeta_{o(s)}^{-j} u_{s,j},$$

where $\mu(K)$ is the group of all the roots of unity in $K$. Then the $K(\mathbf{v})$-algebra $K(\mathbf{v})\mathcal{H}(W)$ is split semisimple.
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By “Tits’ deformation theorem”, the specialization $v_{s,j} \mapsto 1$ induces a bijection

$$\text{Irr}(K(\mathbf{v}) \mathcal{H}(W)) \leftrightarrow \text{Irr}(W)$$

$$\chi_{\mathbf{v}} \mapsto \chi$$
Generic Schur elements

The generic Hecke algebra is endowed with a canonical symmetrizing form $t$. We have that

$$
t = \sum_{\chi \in \text{Irr}(W)} \frac{1}{s_\chi} \chi_v,$$

where $s_\chi$ is the Schur element associated to $\chi_v \in \text{Irr}(K(v)\mathcal{H}(W))$. 

Theorem (C.)
Let $\chi \in \text{Irr}(W)$. The Schur element $s_\chi$ is an element of $\mathbb{Z}_{K[v, v^{-1}]}$ whose irreducible factors (in $K[v, v^{-1}]$) are of the form: $\Psi(M)$ where $\Psi$ is a $K$-cyclotomic polynomial in one variable, $M$ is a primitive monomial of degree 0, i.e., if $M = \prod s_j v^{a_j}$, then $\gcd(a_j, s_j) = 1$ and $\sum s_j v^{a_j} = 0$. 

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where \( s_{\chi} \) is the Schur element associated to \( \chi_{\nu} \in \text{Irr}(K(\nu) \mathcal{H}(W)) \).

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Schur elements of $G_2$: $X_0^2 := u_0$, $X_1^2 := -u_1$, $Y_0^2 := w_0$, $Y_1^2 := -w_1$.

\[ s_1 = \Phi_4(X_0 X_1^{-1}) \cdot \Phi_4(Y_0 Y_1^{-1}) \cdot \Phi_3(X_0 Y_0 X_1^{-1} Y_1^{-1}) \cdot \Phi_6(X_0 Y_0 X_1^{-1} Y_1^{-1}) \]

\[ s_2 = 2 \cdot X_1^2 X_0^{-2} \cdot \Phi_3(X_0 Y_0 X_1^{-1} Y_1^{-1}) \cdot \Phi_6(X_0 Y_1 X_1^{-1} Y_0^{-1}) \]

\[ \Phi_4(x) = x^2 + 1, \quad \Phi_3(x) = x^2 + x + 1, \quad \Phi_6(x) = x^2 - x + 1. \]
Cyclotomic Hecke algebras

Let $y$ be an indeterminate. A cyclotomic specialization of $\mathcal{H}$ is a $\mathbb{Z}_K$-algebra morphism $\phi : \mathbb{Z}_K[v, v^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ of the form:

$$\phi : v_{s,j} \mapsto y^{n_{s,j}}$$

where $n_{s,j} \in \mathbb{Z}$ for all $s$ and $j$. 

Proposition (C.)

The algebra $\mathbb{K}(y)H_\phi$ is split semisimple.
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The corresponding cyclotomic Hecke algebra $\mathcal{H}_\phi$ is the $\mathbb{Z}_K[y, y^{-1}]$-algebra obtained as the specialization of $\mathcal{H}(W)$ via the morphism $\phi$. 

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If $q := y^{|\mu(K)|}$, then the morphism $\phi$ can be also described as follows:

$$\phi : u_{s,j} \mapsto \zeta_j^{o(s)} q^{n_{s,j}}.$$
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The algebra $K(y)\mathcal{H}_\phi$ is split semisimple.
By “Tits’ deformation theorem”, we obtain that the specialization \( v_{s,j} \mapsto 1 \) induces the following bijections:

\[
\text{Irr}(K(v)H) \leftrightarrow \text{Irr}(K(y)H_{\phi}) \leftrightarrow \text{Irr}(W)
\]

\[
\chi_v \mapsto \chi_{\phi} \mapsto \chi
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$$
\begin{align*}
\text{Irr}(K(\nu)\mathcal{H}) & \leftrightarrow \text{Irr}(K(y)\mathcal{H}_\phi) & \leftrightarrow \text{Irr}(W) \\
\chi_{\nu} & \mapsto \chi_\phi & \mapsto \chi
\end{align*}
$$

**Proposition**

The Schur element $s_{\chi_\phi}(y)$ associated to the irreducible character $\chi_\phi$ of $K(y)\mathcal{H}_\phi$ is a Laurent polynomial in $y$ of the form

$$
s_{\chi_\phi}(y) = \psi_{\chi_\phi} y^{a_{\chi_\phi}} \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi_\phi,\Phi}},
$$

where $\psi_{\chi_\phi} \in \mathbb{Z}_K$, $a_{\chi_\phi} \in \mathbb{Z}$, $n_{\chi_\phi,\Phi} \in \mathbb{N}$ and $C_K$ is a set of $K$-cyclotomic polynomials.
Rouquier blocks of $\mathcal{H}_\phi$

The Rouquier blocks of the cyclotomic Hecke algebra $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi$, where

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i.e., the partition $\mathcal{RB}(\mathcal{H}_\phi)$ of $\text{Irr}(\mathcal{W})$ minimal for the property:

For all $B \in \mathcal{RB}(\mathcal{H}_\phi)$ and $h \in \mathcal{H}_\phi$, $\sum_{\chi \in B} \frac{\chi_\phi(h)}{s_{\chi_\phi}} \in \mathcal{R}_K(y)$. 

Weyl group: Rouquier blocks \equiv families of characters

W c.r.g (non-Weyl) : Rouquier blocks \equiv "families of characters"

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**Weyl group** : Rouquier blocks $\equiv$ families of characters
Rouquier blocks of $\mathcal{H}_\phi$

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$W$ Weyl group : Rouquier blocks $\equiv$ families of characters

$W$ c.r.g. (non-Weyl) : Rouquier blocks $\equiv$ “families of characters”
Essential monomials and essential hyperplanes

Let $p$ be a prime ideal of $\mathbb{Z}_K$. 
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A primitive monomial \( M \) in \( \mathbb{Z}_K[v, v^{-1}] \) is called \( p \)-essential for \( W \) if there exist an irreducible character \( \chi \) of \( W \) and a \( K \)-cyclotomic polynomial \( \Psi \) such that

\[
\Psi(M) \mid s \chi(v)
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$$\Psi(M) \text{ divides } s_\chi(v).$$
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2. $\Psi(1) \in p$. 

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A primitive monomial $M$ is called essential for $W$ if it is $p$-essential for some prime ideal $p$ of $\mathbb{Z}_K$. 

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Schur elements of $G_2$ : 2-essential in purple, 3-essential in green.

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\[
\begin{align*}
\Phi_4(x) &= x^2 + 1, & \Phi_3(x) &= x^2 + x + 1, & \Phi_6(x) &= x^2 - x + 1. \\
\Phi_4(1) &= 2, & \Phi_3(1) &= 3, & \Phi_6(1) &= 1.
\end{align*}
\]
Let $\phi : v_{s,j} \mapsto y^{n_{s,j}}$ be a cyclotomic specialization and let $M = \prod_{s,j} v_{s,j}^{a_{s,j}}$ be an essential monomial for $W$. 

Theorem (C.) 

Let $\phi : v_{s,j} \mapsto y^{n_{s,j}}$ be a cyclotomic specialization. The Rouquier blocks of $H_\phi$ is a partition generated by the Rouquier blocks associated with the essential hyperplanes that the $n_{s,j}$ belong to.
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**Theorem (C.)**

Let $\phi : v_{s,j} \mapsto y^{n_{s,j}}$ be a cyclotomic specialization. The Rouquier blocks of $H_\phi$ is a partition generated by the Rouquier blocks associated with the essential hyperplanes that the $n_{s,j}$ belong to.
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a partition, i.e., a finite decreasing sequence of positive integers:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_h \geq 1.$$
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To each partition $\lambda$ we associate its $\beta$-number, $\beta_\lambda = (\beta_1, \beta_2, \ldots, \beta_h)$, defined by

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**Example:** If \( \lambda = (4, 2, 2, 1) \), then \( \beta_\lambda = (7, 4, 3, 1) \).
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**Example:** If $\lambda = (4, 2, 2, 1)$, then $\beta_\lambda = (7, 4, 3, 1)$.

Let $m \in \mathbb{N}$. The $m$-shifted $\beta$-number of $\lambda$ is the sequence of numbers defined by

$$\beta_\lambda[m] = (\beta_1 + m, \beta_2 + m, \ldots, \beta_h + m, m - 1, m - 2, \ldots, 1, 0).$$
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**Example:** If \( \lambda = (4, 2, 2, 1) \), then \( \beta_\lambda[3] = (10, 7, 6, 4, 2, 1, 0) \).
Let \( d \) be a positive integer. A family of \( d \) partitions \( \lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}) \) is called a \( d \)-partition.
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\[
h^{(a)} := h_{\lambda^{(a)}}, \quad \beta^{(a)} := \beta_{\lambda^{(a)}},
\]

and we have

\[
\lambda^{(a)} = (\lambda_1^{(a)}, \lambda_2^{(a)}, \ldots, \lambda_{h^{(a)}}^{(a)}).
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The integer

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is called the size of \( \lambda \). We also say that \( \lambda \) is a \( d \)-partition of \(|\lambda|\).

From now on, we suppose that we have a given “weight system”, i.e., a family of integers

\[
m := (m^{(0)}, m^{(1)}, \ldots, m^{(d-1)}).
\]
We define the \( m \)-charged height of \( \lambda \) to be the integer

\[
hc_{\lambda} := \max \{ hc^{(a)} | (0 \leq a \leq d - 1) \},
\]

where

\[
hc^{(0)} := h^{(0)} - m^{(0)}, \quad hc^{(1)} := h^{(1)} - m^{(1)}, \ldots, \quad hc^{(d-1)} := h^{(d-1)} - m^{(d-1)}.
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**Definition (m-charged standard symbol and content)**

The $m$-charged standard symbol of $\lambda$ is the family of numbers defined by

$$Bc_\lambda = (Bc^{(0)}_\lambda, Bc^{(1)}_\lambda, \ldots, Bc^{(d-1)}_\lambda),$$

where, for all $a$ ($0 \leq a \leq d - 1$), we have

$$Bc^{(a)}_\lambda := \beta^{(a)}[hc_\lambda - hc^{(a)}].$$
We define the \textit{m-charged height} of \( \lambda \) to be the integer
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\]
where, for all \( a \) (\( 0 \leq a \leq d - 1 \)), we have
\[
B_{c_\lambda}^{(a)} := \beta^{(a)}[hc_\lambda - hc^{(a)}].
\]

The \textit{m-charged content} of \( \lambda \) is the multiset
\[
\text{Cont}_{c_\lambda} = B_{c_\lambda}^{(0)} \cup B_{c_\lambda}^{(1)} \cup \ldots \cup B_{c_\lambda}^{(d-1)}.
\]
Example: Let us take $d = 2$, $\lambda = ((3), (2, 1))$ and $m = (2, -1)$. Then

- $\beta^{(0)} = (3)$,
- $\beta^{(1)} = (3, 1)$,
- $hc^{(0)} = 1 - 2 = -1$,
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Consequently,

$$Bc_\lambda = \begin{pmatrix} 7 & 3 & 2 & 1 & 0 \\ 3 & 1 & & & \end{pmatrix}.$$
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Consequently,

$$Bc_\lambda = \begin{pmatrix} 7 & 3 & 2 & 1 & 0 \\ 3 & 1 \\ & & & & \end{pmatrix}.$$ 

We have $\text{Contc}_\lambda = \{0, 1, 1, 2, 3, 3, 7\}$. 
The group $G(d, 1, r)$

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- $G(d, 1, r) \simeq \mu_d \wr S_r$. 

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- $G(d, 1, r) \cong \mu_d \wr \mathfrak{S}_r$.
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Ariki-Koike algebras

The “generic” Ariki-Koike algebra associated to $G(d, 1, r)$ is the algebra $\mathcal{H}_{d,r}$ generated over the Laurent polynomial ring in $d + 1$ indeterminates

$$\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \ldots, u_{d-1}, u_{d-1}^{-1}, x, x^{-1}]$$

by the elements $s, t_1, t_2, \ldots, t_{r-1}$ satisfying the relations

- $st_1st_1 = t_1st_1s$,
- $st_j = t_j s$, for all $j = 2, \ldots, r - 1$,
- $t_{j-1}t_jt_{j-1} = t_jt_{j-1}t_j$, for all $j = 2, \ldots, r - 1$,
- $t_it_j = t_jt_i$, for all $1 \leq i, j \leq r - 1$ with $|i - j| > 1$,
- $(s - u_0)(s - u_1)\ldots(s - u_{d-1}) = 0$,
- $(t_j - x)(t_j + 1) = 0$, for all $j = 1, \ldots, r - 1$. 
The Schur elements of $\mathcal{H}_{d,r}$ have been calculated independently by Geck, Iancu, Malle (2000), Mathas (2004).

Let $\varphi : \{ u_j \mapsto \zeta^d_j q^m, (0 \leq j < d) \}$ be a cyclotomic specialization for $\mathcal{H}_{d,r}$.

**Proposition (C.)**

The essential hyperplanes for $G(\mathcal{H}_{d,1})$ are given by:

$$N_k N + M_s - M_t = 0 \quad \text{for} \quad -r < k < r \quad \text{and} \quad 0 \leq s < t < d \quad \text{such that} \quad \zeta^s d - \zeta^t d \text{ is not a unit in } \mathbb{Z}[\zeta^d].$$
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Let
\[ \phi : \begin{cases} 
  u_j &\mapsto \zeta_d^j q^{m_j}, \quad (0 \leq j < d), \\
  x &\mapsto q^n 
\end{cases} \]
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The essential hyperplanes for $G(d, 1, r)$ are given by:
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**Proposition (C.)**

The essential hyperplanes for $G(d, 1, r)$ are given by:

- $N = 0$. 

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Rouquier blocks of the “cyclotomic” Ariki-Koike algebras

Proposition
The Rouquier blocks associated with no essential hyperplane are trivial.

In order to obtain a description for the Rouquier blocks associated with the essential hyperplanes of $G(d,1,r)$, we have used the algorithm for the blocks of the Ariki-Koike algebra over a field given by Lyle and Mathas (2007).

Proposition (C.)
Let $\lambda,\mu$ be two $d$-partitions of $r$. The characters $\chi_\lambda$ and $\chi_\mu$ are in the same Rouquier block associated with the essential hyperplane $N = 0$ if and only if $|\lambda(a)| = |\mu(a)|$ for all $a = 0, 1, \ldots, d - 1$. 

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$$|\lambda^{(a)}| = |\mu^{(a)}| \text{ for all } a = 0, 1, \ldots, d - 1.$$
Let $H : kN + M_s - M_t = 0$ be an essential hyperplane for $G(d, 1, r)$ and let

$$\phi : \left\{ \begin{array}{l}
u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\
x \mapsto q^n \end{array} \right.$$ 

be a cyclotomic specialization such that $kn + m_s - m_t = 0$ and the integers $n$ and $m_j \ (0 \leq j < d)$ belong to no other essential hyperplane for $G(d, 1, r)$. 

Theorem (Broué-Kim)

Let $\lambda, \mu$ be two $d$-partitions of $r$. If the irreducible characters $(\chi_\lambda)$ and $(\chi_\mu)$ are in the same Rouquier block of $(H_d, r)$, then

$$\text{Cont}^c_\lambda = \text{Cont}^c_\mu$$

with respect to the weight system $(m_0, m_1, \ldots, m_{d-1})$.

The converse is true when $d$ is a power of a prime number.
Let $H : kN + M_s - M_t = 0$ be an essential hyperplane for $G(d, 1, r)$ and let 

$$\phi : \begin{cases} 
  u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\
  x \mapsto q^n \end{cases}$$

be a cyclotomic specialization such that $kn + m_s - m_t = 0$ and the integers $n$ and $m_j$ ($0 \leq j < d$) belong to no other essential hyperplane for $G(d, 1, r)$. Without loss of generality, we shall assume that $n = 1$. 

Theorem (Broué-Kim) Let $\lambda, \mu$ be two $d$-partitions of $r$. If the irreducible characters $(\chi_{\lambda})_{\phi}$ and $(\chi_{\mu})_{\phi}$ are in the same Rouquier block of $(\mathcal{H}_d, r)$, then $\text{Contc}_{\lambda} = \text{Contc}_{\mu}$ with respect to the weight system $(m_0, m_1, \ldots, m_{d-1})$. 

The converse is true when $d$ is a power of a prime number.
Let \( H : kN + M_s - M_t = 0 \) be an essential hyperplane for \( G(d, 1, r) \) and let

\[
\phi : \left\{ \begin{array}{l}
  u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\
  x \mapsto q^n
\end{array} \right.
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**Theorem (Broué-Kim)**

Let \( \lambda, \mu \) be two \( d \)-partitions of \( r \). If the irreducible characters \( (\chi_\lambda)_\phi \) and \( (\chi_\mu)_\phi \) are in the same Rouquier block of \( (\mathcal{H}_{d,r})_\phi \), then \( \text{Contc}_{\lambda} = \text{Contc}_{\mu} \) with respect to the weight system \( (m_0, m_1, \ldots, m_{d-1}) \).
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**Theorem (Broué-Kim)**

Let $\lambda, \mu$ be two $d$-partitions of $r$. If the irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, then $\text{Cont}_{c\lambda} = \text{Cont}_{c\mu}$ with respect to the weight system $(m_0, m_1, \ldots, m_{d-1})$. The converse is true when $d$ is a power of a prime number.
Proposition (C.)

Let $\lambda, \mu$ be two $d$-partitions of $r$. The irreducible characters $(\chi_{\lambda})_{\phi}$ and $(\chi_{\mu})_{\phi}$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_{\phi}$ if and only if:

1. We have $\lambda(a) = \mu(a)$ for all $a \in \{s, t\}$.
2. If $\lambda_{st} := (\lambda(s), \lambda(t))$ and $\mu_{st} := (\mu(s), \mu(t))$, then $\text{Contc}_{\lambda_{st}} = \text{Contc}_{\mu_{st}}$ with respect to the weight system $(m_s, m_t)$.

Let $\lambda_{st}$ and $\mu_{st}$ be as above and set $l := |\lambda_{st}| = |\mu_{st}|$.

Let us consider the Ariki-Koike algebra $\mathcal{H}_{2,l}$ of $G(2,1,l)$ over the Laurent polynomial ring $\mathbb{Z}[U_0, U_{-1}, U_1, U_{-1}, X, X_{-1}]$ and the cyclotomic specialization $\vartheta: U_0 \mapsto q^{m_s}, U_1 \mapsto -q^{m_t}, X \mapsto q^n$.

By the theorem of Broué-Kim, we have $\text{Contc}_{\lambda_{st}} = \text{Contc}_{\mu_{st}}$ with respect to the weight system $(m_s, m_t)$ if and only if the corresponding characters of $G(2,1,l)$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_{\vartheta}$. 
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Let \( \lambda^{st} \) and \( \mu^{st} \) be as above and set \( l := |\lambda^{st}| = |\mu^{st}|. \) Let us consider the Ariki-Koike algebra \( \mathcal{H}_{2,l} \) of \( G(2, 1, l) \) over the Laurent polynomial ring

\[
\mathbb{Z}[U_0, U_0^{-1}, U_1, U_1^{-1}, X, X^{-1}]
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$$\mathbb{Z}[U_0, U_0^{-1}, U_1, U_1^{-1}, X, X^{-1}]$$

and the cyclotomic specialization

$$\vartheta : U_0 \mapsto q^{m_s}, U_1 \mapsto -q^{m_t}, X \mapsto q^n.$$
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Let $\lambda^{st}$ and $\mu^{st}$ be as above and set $l := |\lambda^{st}| = |\mu^{st}|$. Let us consider the Ariki-Koike algebra $\mathcal{H}_{2,l}$ of $G(2,1,l)$ over the Laurent polynomial ring

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By the theorem of Broué-Kim, we have $\text{Contc}_{\lambda^{st}} = \text{Contc}_{\mu^{st}}$ with respect to the weight system $(m_s, m_t)$ if and only if the corresponding characters of $G(2,1,l)$ belong to the same Rouquier block of $(\mathcal{H}_{2,l})_\vartheta$. 
Example: Let $W := G(3, 1, 2)$. 
**Example:** Let $W := G(3, 1, 2)$. The irreducible characters of $W$ are parametrized by the 3-partitions of 2. These are:
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\[ \lambda_{(2),0} = ((2), \emptyset, \emptyset), \quad \lambda_{(2),1} = (\emptyset, (2), \emptyset), \quad \lambda_{(2),2} = (\emptyset, \emptyset, (2)), \]
\[ \lambda_{(1,1),0} = ((1, 1), \emptyset, \emptyset), \quad \lambda_{(1,1),1} = (\emptyset, (1, 1), \emptyset), \quad \lambda_{(1,1),2} = (\emptyset, \emptyset, (1, 1)), \]
\[ \lambda_{\emptyset,0} = (\emptyset, (1), (1)), \quad \lambda_{\emptyset,1} = ((1), \emptyset, (1)), \quad \lambda_{\emptyset,2} = ((1), (1), \emptyset). \]
Example: Let $W := G(3, 1, 2)$. The irreducible characters of $W$ are parametrized by the 3-partitions of 2. These are:

$$
\begin{align*}
\lambda_{(2),0} &= ((2), \emptyset, \emptyset), & \lambda_{(2),1} &= (\emptyset, (2), \emptyset), & \lambda_{(2),2} &= (\emptyset, \emptyset, (2)), \\
\lambda_{(1,1),0} &= ((1, 1), \emptyset, \emptyset), & \lambda_{(1,1),1} &= (\emptyset, (1, 1), \emptyset), & \lambda_{(1,1),2} &= (\emptyset, \emptyset, (1, 1)), \\
\lambda_{\emptyset,0} &= (\emptyset, (1), (1)), & \lambda_{\emptyset,1} &= ((1), \emptyset, (1)), & \lambda_{\emptyset,2} &= ((1), (1), \emptyset).
\end{align*}
$$

The generic Ariki-Koike algebra associated to $W$ is the algebra $\mathcal{H}_{3,2}$ generated over the Laurent polynomial ring in 4 indeterminates

$$\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, u_2, u_2^{-1}, x, x^{-1}]$$

by the elements $s$ and $t$ satisfying the relations

- $ssts = tsts$,
- $(s - u_0)(s - u_1)(s - u_2) = (t - x)(t + 1) = 0$. 
Let
\[ \phi : \begin{cases} 
  u_j \mapsto \zeta_3^j q^{m_j}, (0 \leq j \leq 2), \\
  x \mapsto q^n 
\end{cases} \]
be a cyclotomic specialization for \( H_{3,2} \).
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\end{cases}
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be a cyclotomic specialization for \( \mathcal{H}_{3,2} \). The essential hyperplanes for \( W \) are:

1. \( kN + M_0 - M_1 = 0 \) for \( k \in \{-1, 0, 1\} \).
2. \( kN + M_0 - M_2 = 0 \) for \( k \in \{-1, 0, 1\} \).
3. \( kN + M_1 - M_2 = 0 \) for \( k \in \{-1, 0, 1\} \).

Let us take \( m_0 = 0, m_1 = 0, m_2 = 5 \) and \( n = 1 \). These integers belong only to the essential hyperplane \( M_0 - M_1 = 0 \).

Following our main result, two irreducible characters (\( \chi_{\lambda}^{\phi} \), \( \chi_{\mu}^{\phi} \)) are in the same Rouquier block of (\( H_{2,3}^{\phi} \)) if and only if:

1. \( \lambda^{(2)} = \mu^{(2)} \).
2. If \( \lambda_{01} = (\lambda^{(0)}, \lambda^{(1)}) \) and \( \mu_{01} = (\mu^{(0)}, \mu^{(1)}) \), then \( \text{Cont}_c \lambda_{01} = \text{Cont}_c \mu_{01} \) with respect to the weight system \((0, 0)\).
Let
\[ \phi : \{ \begin{array}{c} u_j \mapsto \zeta_3^j q^{m_j}, (0 \leq j \leq 2), \\ x \mapsto q^n \end{array} \] 
be a cyclotomic specialization for \( H_{3,2} \). The essential hyperplanes for \( W \) are:

- \( N = 0 \).
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- \( KN + M_1 - M_2 = 0 \) for \( k \in \{-1, 0, 1\} \).
Let

\[ \phi : \begin{cases} 
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Let us take \( m_0 := 0, \ m_1 := 0, \ m_2 := 5 \) and \( n := 1 \).
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Maria Chlouveraki (EPFL) 
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\[ B_{\lambda_{(1,1),0}} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \]

\[ B_{\lambda_{\emptyset,0}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_{\lambda_{\emptyset,1}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]
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\]

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\]

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The Rouquier blocks of $(\mathcal{H}_{3,2})_{\phi}$ are:
Consequently, the characters corresponding to the partitions $\lambda_{(2),2}$, $\lambda_{(1,1),2}$ and $\lambda_{0,2}$ are singletons. Moreover, we have

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The Rouquier blocks of $(\mathcal{H}_{3,2})_\phi$ are:

$$\{ \lambda_{(2),0}, \lambda_{(2),1} \}, \{ \lambda_{(2),2} \}, \{ \lambda_{(1,1),0}, \lambda_{(1,1),1} \}, \{ \lambda_{(1,1),2} \}, \{ \lambda_{0,0}, \lambda_{0,1} \}, \{ \lambda_{0,2} \}.$$
The group $G(de, e, r)$

The group $G(de, e, r)$ is the group of all $r \times r$ monomial matrices with non-zero entries in $\mu_{de}$ and product of the non-zero entries in $\mu_d$. 
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- Thanks to a result by Ariki (1995), any cyclotomic Hecke algebra of $G(de, e, r)$, $r > 2$, can be viewed as a subalgebra of a cyclotomic Hecke algebra associated to $G(de, 1, r)$. Then Clifford Theory allows us to obtain the Rouquier blocks of the former from the Rouquier blocks of the latter.
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- The same holds in the case where $r = 2$ and $e$ is odd.
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- The same holds in the case where $r = 2$ and $e$ is odd.

- In the case where $r = 2$ and $e$ is even, explicit calculations had to be made (and there is no combinatorial description of the Rouquier blocks).