The irreducible representations of B_3 of dimension at most 5

Eirini Chavli

Université Paris Diderot

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Braids

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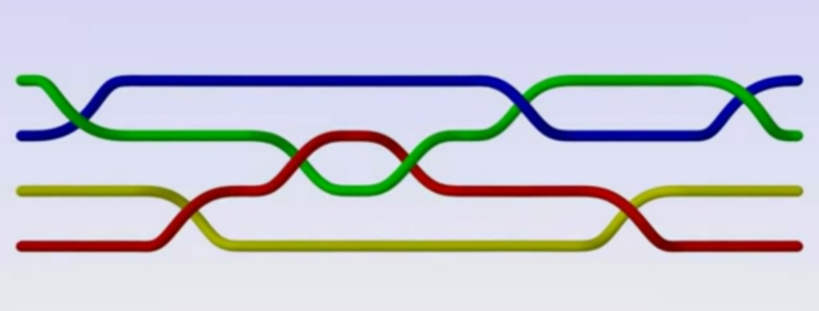

The braid group B_3

Theorem (Artin 1928)

The braid group B_n on n strands admits a presentation with generators

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and relations

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The first "new" example is B_3.

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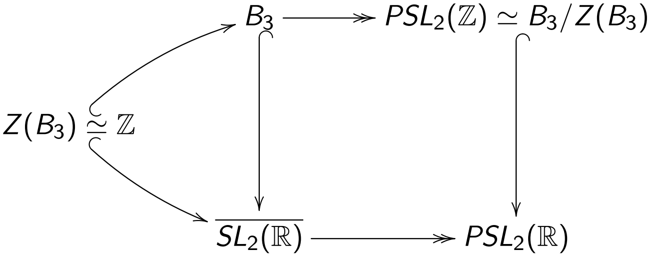
- B_3 is the universal central extension of the modular group $PSL_2(\mathbb{Z})$



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If $\sigma_1 \mapsto A, \sigma_2 \mapsto B$ is such a representation, we have:

- A and B can be chosen of order triangular form.
- The coefficients of A and B are completely determined by the eigenvalues (for $k = 2, 3$) or by the eigenvalues and by a choice of a kth root of $\det A$ that we call r (for $k = 4, 5$).
- Such irreducible representations exist if the eigenvalues do not annihilate some polynomials P_k in eigenvalues and r.

Example: The irreducible representations of dimension 3

$$A = \begin{bmatrix} \lambda_3 & 0 & 0 \\ \lambda_1\lambda_3 + \lambda_2^2 & \lambda_2 & 0 \\ \lambda_2 & 1 & \lambda_1 \end{bmatrix}, \; B = \begin{bmatrix} \lambda_1 & -1 & \lambda_2 \\ 0 & \lambda_2 & -\lambda_1\lambda_3 - \lambda_2^2 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

where $(\lambda_1^2 + \lambda_2\lambda_3)(\lambda_2^2 + \lambda_1\lambda_3)(\lambda_3^2 + \lambda_1\lambda_2) \neq 0$.

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- What is the nature of the polynomials P_k and why do they provide a necessary condition for a representation of this form to be irreducible?

We will answer these questions by recovering this classification as a consequence of the freeness conjecture for the generic Hecke algebra of the finite quotients of the braid group B_3 defined by the additional relation $\sigma_i^k = 1$, for $i = 1, 2$.

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- $n = 1$. Then $W = \{1\}$.
- $n = 2$. Then, $W = \mathbb{Z}/k\mathbb{Z}$.
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- $k = 2$. Then, $W = S_n$.
- $n = 3$ and $k = 3, 4, 5$. Then $W = G_4,\ G_8,\ G_{16}$.
- $n = 4$ and $k = 3$. Then $W = G_{25}$.
- $n = 5$ and $k = 3$. Then $W = G_{32}$.

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Theorem (Broué-Malle, C., Marin)

H_W is a free R-module of rank $|W|$.

The irreducible representations of B_3 of dimension $k \leq 5$

Let $\rho : \; B_3 \rightarrow \; GL_k(\mathbb{C})$ be a representation of B_3 of dimension $k \leq 5$ over $\mathbb{C}$. Let $A := \rho(\sigma_1)$ and $B := \rho(\sigma_2)$, with eigenvalues $\lambda_1, \ldots, \lambda_k$.

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By the theorem of Cayley-Hamilton there is a monic polynomial $Q(X) = (X - \lambda_1) \ldots (X - \lambda_k) \in \mathbb{C}[X]$ of degree k such that $Q(A) = Q(B) = 0$.

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We only need to describe the irreducible $\mathbb{C}H_k := H_k \otimes_\theta \mathbb{C}$-modules of dimension k.

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where s_χ is the Schur element of χ with respect to t.

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Let $R_0(KH_k)$ be the Grothendieck group of finite-dimensional KH_k-modules.

Let $R_0^+(KH_k)$ be the subset of $R_0(KH_k)$ consisting of elements $[V]$, where V is a finite-dimensional KH_k-module.

We can define the decomposition map d_θ associated to θ.

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Proposition (C. 2014)

There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

The decomposition matrix

The corresponding decomposition matrix is the $\mathrm{Irr}(KH_k)\times$ $\mathrm{Irr}(\mathbb{C}H_k)$ matrix $(d_{\chi\phi})$ with non-negative integer entries such that $d_\theta([V_\chi]) = \sum_\phi d_{\chi\phi}[V'_\phi]$.

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This matrix records in which way the irreducible representations of the semisimple algebra KH_k break up into irreducible representations of $\mathbb{C}H_k$.

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There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 2$.

$$\begin{array}{c} \begin{array}{ccc} \dots & \phi_S & \dots \end{array} \\ \left(\begin{array}{ccc} & 0 & \\ & 0 & \\ 0 & 1 & 0 \end{array}\right) \begin{array}{l} \chi_{1,1} \\ \chi_{1,2} \\ \chi_{2,1} \end{array} \end{array}$$

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 2$.

$$\begin{array}{c} \begin{array}{ccc} \dots & \phi_S & \dots \end{array} \\ \left(\begin{array}{ccc} & 0 & \\ & 0 & \\ 0 & 1 & 0 \end{array}\right) \begin{array}{c} \chi_{1,1} \\ \chi_{1,2} \\ \chi_{2,1} \end{array} \end{array}$$

Then, $d_\theta([M]) = [S]$, where M is the 2-dimensional KH_2 module.

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 2$.

$$\begin{array}{ccc|c} \dots & \phi_S & \dots & \\ \end{array}$$
$$\begin{pmatrix} & 0 & \\ & 0 & \\ 0 & 1 & 0 \end{pmatrix} \begin{matrix} \chi_{1,1} \\ \chi_{1,2} \\ \chi_{2,1} \end{matrix}$$

Then, $d_\theta([M]) = [S]$, where M is the 2-dimensional KH_2 module.
Moreover, $\theta(s_{\chi_{2,1}}) \neq 0$.

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 3$.

$$\begin{array}{c} \begin{array}{ccc} \cdots & \phi_S & \cdots \end{array} \\ \left(\begin{array}{ccc} & 0 & \\ & 0 & \\ & 0 & \\ & 0 & \\ & 0 & \\ & 0 & \\ 0 & 1 & 0 \end{array}\right) \begin{array}{l} \chi_{1,1} \\ \chi_{1,2} \\ \chi_{1,3} \\ \chi_{2,1} \\ \chi_{2,2} \\ \chi_{2,3} \\ \chi_{3,1} \end{array} \end{array}$$

Then, $d_\theta([M]) = [S]$, where M is the 3-dimensional KH_3 module.
Moreover, $\theta(s_{\chi_{3,1}}) \neq 0$.

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 4$.

$$\begin{array}{ccc} \dots & \phi_S & \dots \end{array}$$

$$\left(\begin{array}{ccc} & 0 & \\ & 0 & \\ & 0 & \\ & & \\ & & \end{array}\right) \begin{array}{l} \chi_{1,i} \\ \chi_{2,j} \\ \chi_{3,k} \\ \chi_{4,1} \\ \chi_{4,2} \end{array}$$

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 4$.

$$\begin{array}{ccc|c} \dots & \phi_S & \dots & \\ & 0 & & \chi_{1,i} \\ & 0 & & \chi_{2,j} \\ & 0 & & \chi_{3,l} \\ 0 & 1 & 0 & \chi_{4,1} \\ & 0 & & \chi_{4,2} \end{array}$$

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 4$.

$$\begin{array}{cccc} \dots & \phi_S & \dots & \\ \left(\begin{array}{c} \\ \\ \\ \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} & \left.\begin{array}{c} \\ \\ \\ \\ 0 \end{array}\right) & \begin{array}{c} \chi_{1,i} \\ \chi_{2,j} \\ \chi_{3,l} \\ \chi_{4,1} \\ \chi_{4,2} \end{array} \end{array}$$

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 4$.

$$\begin{array}{ccc} \dots & \phi_S & \dots \end{array}$$
$$\begin{pmatrix} & 0 & \\ & 0 & \\ & 0 & \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{array}{c} \chi_{1,i} \\ \chi_{2,j} \\ \chi_{3,l} \\ \chi_{4,1} \\ \chi_{4,2} \end{array}$$

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 4$.

$$\begin{array}{ccc} \cdots & \phi_S & \cdots \end{array}$$
$$\begin{pmatrix} & 0 & \\ & 0 & \\ & 0 & \\ & 0 & \\ 0 & 1 & 0 \end{pmatrix} \begin{array}{l} \chi_{1,i} \\ \chi_{2,j} \\ \chi_{3,l} \\ \chi_{4,a} \\ \chi_{4,b} \end{array}$$

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 4$.

$$
\begin{array}{c}
\begin{array}{ccc} \cdots & \phi_S & \cdots \end{array} \\
\left(\begin{array}{ccc} & 0 & \\ & 0 & \\ & 0 & \\ & 0 & \\ 0 & 1 & 0 \end{array}\right)
\begin{array}{l} \chi_{1,i} \\ \chi_{2,j} \\ \chi_{3,l} \\ \chi_{4,a} \\ \chi_{4,b} \end{array}
\end{array}
$$

Then, $d_\theta([M]) = [S]$, where M is one of the two 4-dimensional KH_4 modules.
Moreover, $\theta(s_{\chi_{4,b}}) \neq 0$.

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 5$.

$$\begin{array}{c} \begin{array}{ccc} \dots & \phi_S & \dots \end{array} \\ \left(\begin{array}{ccc} & 0 & \\ & \vdots & \\ & 0 & \\ & & \\ & & \end{array} \right) \begin{array}{c} \\ \\ \\ \chi_{5,i} \\ \chi_{6,j} \end{array} \end{array}$$

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 5$.

$$\begin{array}{c} \begin{array}{ccc} \dots & \phi_S & \dots \end{array} \\ \left(\begin{array}{ccc} & 0 & \\ & \vdots & \\ & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right) \begin{array}{l} \\ \\ \\ \chi_{5,i} \\ \chi_{6,j} \end{array} \end{array}$$

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible $\mathbb{C}(\mathbf{v})H_k$-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 5$.

$$\begin{array}{c} \begin{array}{ccc} \phi_J & \phi_S & \cdots \end{array} \\ \left(\begin{array}{ccc} & \vdots & \\ & 0 & \\ & & \\ 1 & 1 & 0 \end{array} \right) \begin{array}{l} \\ \\ \chi_{5,i} \\ \chi_{6,j} \end{array} \end{array}$$

J is a 1-dimensional KH_5-module.

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible $\mathbb{C}(\mathbf{v})H_k$-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 5$.

$$
\begin{array}{c}
\begin{matrix} \phi_J & \phi_S & \cdots \end{matrix} \\
\begin{pmatrix} 1 & & \\ & \vdots & \\ & 0 & \\ & & \\ 1 & 1 & 0 \end{pmatrix}
\begin{matrix} \chi_1 \\ \\ \\ \chi_{5,i} \\ \chi_{6,j} \end{matrix}
\end{array}
$$

J is a 1-dimensional KH_5-module.

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible $\mathbb{C}(\mathbf{v})H_k$-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 5$.

$$\begin{array}{c} \begin{matrix} \phi_J & \phi_S & \dots \end{matrix} \\ \begin{pmatrix} 1 & & \\ & \vdots & \\ & 0 & \\ & & \\ 1 & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} \chi_1 \\ \\ \\ \chi_{5,i} \\ \chi_{6,j} \end{matrix}$$

J is a 1-dimensional KH_5-module.
We assume that $\det A \neq -\lambda_i^6 \lambda_j^{-1}$.

The irreducible representations of B_3 of dimension k

Let S be an irreducible $\mathbb{C}H_k$-module of dimension k.
There is an irreducible KH_k-module M such that $d_\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- $k = 5$.

$$\begin{array}{c} \begin{array}{ccc} \cdots & \phi_S & \cdots \end{array} \\ \left(\begin{array}{ccc} & 0 & \\ & \vdots & \\ & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array}\right) \begin{array}{l} \\ \\ \\ \chi_{5,i} \\ \chi_{6,j} \end{array} \end{array}$$

Then, $d_\theta([M]) = [S]$, where M is one of the five 5-dimensional KH_5 modules.
Moreover, $\theta(s_{\chi_{5,i}}) \neq 0$.

Last slide

Thank You!