A new 2-variable generalization of the Jones polynomial

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What is a knot?

Definition

A knot K is the embedding of S^1 into S^3 or $\mathbb{R}^3$.

Knot equivalence

Two knots, K_1 and K_2, are considered equivalent if there exists a continuous deformation that takes us from K_1 to K_2.

Definition

Two knots are equivalent if there exists an ambient isotopy between them. That is, if there exists:

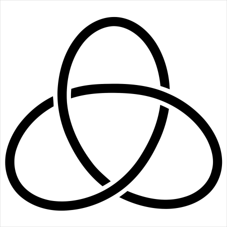
$$f_t : S^3 \longrightarrow S^3, \quad t \in [0,1]$$

such that: f_0 is the identity map in S^3 and $f_1(K_1) = K_2$.

Knot diagrams

Definition

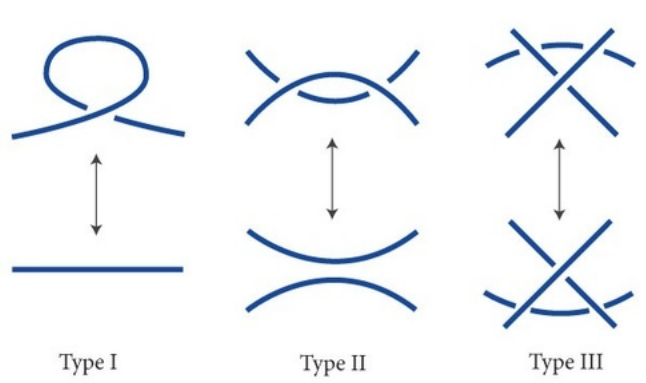
A knot diagram is a picture of a projection of a knot onto a plane, where only double points are allowed (no more than two points are allowed to be superposed). It is usually demanded that a knot diagram contain the information if the crossings are overcrossings or undercrossings so that the original knot can be reconstructed.



Reidemeister moves

Theorem (Reidemeister 1935)

Two knots K_1 and K_2 are isotopic if and only if their corresponding diagrams are related by a finite sequence of the following moves:



Knot Invariants

Let $\mathcal{L}$ be the set of all knots and all links.

Definition

A knot (link) invariant is a map:

$$\mathrm{I} : \mathcal{L} \longrightarrow \mathcal{S}$$

such that:

$$\mathrm{K}_1 \sim \mathrm{K}_2 \Longrightarrow \mathrm{I}(\mathrm{K}_1) = \mathrm{I}(\mathrm{K}_2)$$

or, equivalently:

$$\mathrm{I}(\mathrm{K}_1) \neq \mathrm{I}(\mathrm{K}_2) \Longrightarrow \mathrm{K}_1 \nsim \mathrm{K}_2$$

The most famous knot invariant is the Jones polynomial (1984).

- Use of a representation of the braid group that satisfies a particular quadratic relation.

The Hecke algebra of type A

$$H_n(q) = \mathrm{Alg}_{\mathbb{C}} \left\{ h_1, \ldots, h_{n-1} \middle| \begin{array}{cc} h_i h_j = h_j h_i & |i-j| > 1 \\ h_i h_j h_i = h_j h_i h_j & |i-j| = 1 \\ h_i^2 = 1 + (q - q^{-1}) h_i & \end{array} \right\}$$

- There exists an epimorphism from $\mathbb{C}B_n$ to $H_n(q)$ sending $\sigma_i \mapsto h_i$.

Theorem (Ocneanu 1984)

For any $\zeta \in \mathbb{C}^\times$ there exists a unique linear function

$$\tau : \cup_\infty H_n(q) \longrightarrow \mathbb{C}[\zeta]$$

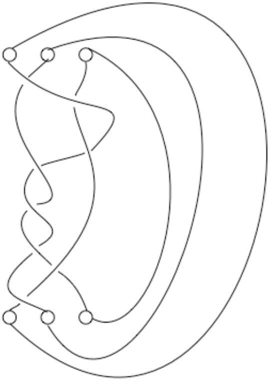
that can be inductively defined by the following rules:

1. $\tau(1) = 1$
2. $\tau(ab) = \tau(ba) \quad a, b \in H_n(q)$
3. $\tau(ah_n) = \zeta\tau(a) \quad a \in H_n(q)$

Link invariants through τ

Theorem (Alexander 1923)

Any oriented link is isotopic to the closure of a braid.



Link invariants through τ

Isotopy classes of links are in bijection with equivalence classes of braids under:

Markov Moves

I. Conjugation: $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in B_n$

II. Stabilization moves: $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n$

Markov trace

2. Conjugation: $\tau(ab) = \tau(ba)$
3. Markov Property: $\tau(ah_n) = \zeta\tau(a)$

   $a, b \in H_n(u)$.

The Homflypt polynomial

Re-scaling according to the Markov braid equivalence and normalizing the trace τ we obtain:

Definition (Homflypt polynomial, 1985)

The two–variable invariant $\mathrm{P}(\mathrm{q}, \lambda_{\mathrm{H}})$ of the oriented link L is the function:

$$\mathrm{P}(\mathrm{q}, \lambda_{\mathrm{H}})(\widehat{\alpha}) = \left(\frac{1 - \lambda_{\mathrm{H}}}{\sqrt{\lambda_{\mathrm{H}}}(\mathrm{q} - \mathrm{q}^{-1})} \right)^{\mathrm{n}-1} \left(\sqrt{\lambda_{\mathrm{H}}} \right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)),$$

where $\alpha \in \mathrm{B}_{\mathrm{n}}$, $\varepsilon(\alpha)$ is the algebraic sum of the exponents of the σ_{i}'s in α and π is the natural epimorphism of $\mathbb{C}\mathrm{B}_{\mathrm{n}}$ onto $\mathrm{H}_{\mathrm{n}}(\mathrm{q})$.

The Temperley-Lieb algebra and the Jones polynomial

Definition

For $n \geq 3$, the Temperley-Lieb algebra $TL_n(q)$ is defined as the quotient of the algebra $H_n(q)$ over the 2-sided ideal generated by the element

$$1 + q(h_1 + h_2) + q^2(h_1 h_2 + h_2 h_1) + q^3 h_1 h_2 h_1$$

$$V(q)(\widehat{\alpha}) := \left(-\frac{1+q^2}{q}\right)^{n-1} q^{2\varepsilon(\alpha)} \tau\left(\pi(\alpha)\right) = P(q, q^4)(\widehat{\alpha}),$$

What is (modular) Framization?

- Proposed by Juyumaya and Lambropoulou.
- The addition of framing generators to a knot algebra.
- A new algebra is obtained which is related to framed knots.

The (modular) framed braid group

The framed braid group on n strands $\mathcal{F}_n := \mathbb{Z}^n \rtimes B_n$

Generators of $\mathcal{F}_n$		
	$t_i := (0, \ldots, 0, 1, 0, \ldots, 0)$	framing generators
	σ_i	braiding generators

$$\mathcal{F}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1}, t_1, \ldots, t_n \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j| = 1 \\ t_i t_j = t_j t_i \quad \sigma_i t_j = t_{s_i(j)}\sigma_i & \end{array} \right\rangle$$

If we consider framings modulo d ($t_i^d = 1$) in the above presentation we obtain the modular framed braid group $\mathcal{F}_{d,n}$.

Diagrammatic Interpretation of framed braids

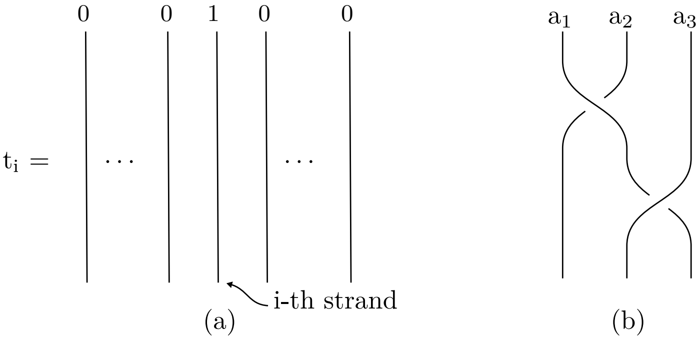


Figure: (a) The generator t_i in $\mathcal{F}_n$ and (b) a framed braid in $\mathcal{F}_3$

The Yokonuma–Hecke algebra

$$Y_{d,n}(q) = \mathrm{Alg}_{\mathbb{C}} \left\{ g_1, \ldots, g_{n-1}, t_1, \ldots, t_n \;\middle|\; \begin{array}{ll} g_i g_j = g_j g_i & |i-j| > 1 \\ g_i g_j g_i = g_j g_i g_j & |i-j| = 1 \\ t_i^d = 1, \quad t_i t_j = t_j t_i & \\ g_i t_j = t_{s_i(j)} g_i & \\ g_i^2 = 1 + (q - q^{-1}) e_i g_i & \end{array} \right\}$$

where $e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}$.

- There exists a natural epimorphism from $\mathcal{F}_{d,n}$ to $Y_{d,n}(q)$ sending:

$$\sigma_i \mapsto g_i \quad \text{and} \quad t_i \mapsto t_i.$$

- For $d = 1$ $Y_{d,n}(q)$ coincides with Iwahori-Hecke algebra.
- It is the basic example of framization of a knot algebra.

N. Thiem, and M. Chlouveraki & L. Poulain d'Andecy studied the Representation Theory of the Yokonuma-Hecke algebra.

Diagrammatic interpretations

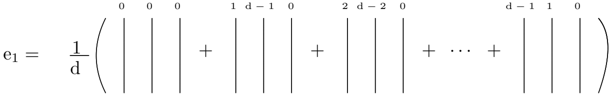


Figure: The element $e_1 \in \mathbb{C}\mathcal{F}_{d,3}$.

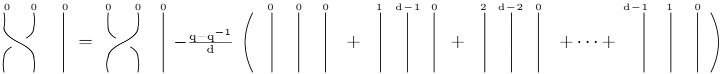


Figure: The element $g_1^{-1} \in Y_{d,3}(u)$.

Markov trace on $\mathrm{Y}_{\mathrm{d,n}}(\mathrm{q})$ (Juyumaya, 2004)

Let d a positive integer. For indeterminates z, $\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{d}-1}$ there exists a unique linear Markov trace tr_d:

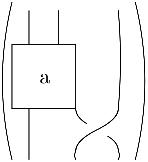
$$\mathrm{tr}_\mathrm{d} : \cup_{\mathrm{n}=1}^{\infty} \mathrm{Y}_{\mathrm{d,n}}(\mathrm{q}) \longrightarrow \mathbb{C}[\mathrm{z}, \mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{d}-1}]$$

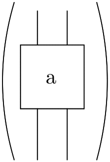
defined inductively on n by the following rules:

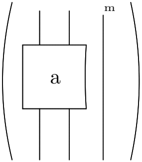
$$\begin{array}{rcll} \mathrm{tr}_\mathrm{d}(\mathrm{ab}) & = & \mathrm{tr}_\mathrm{d}(\mathrm{ba}) & \\ \mathrm{tr}_\mathrm{d}(1) & = & 1 & \\ \mathrm{tr}_\mathrm{d}(\mathrm{a}\mathrm{g}_\mathrm{n}) & = & \mathrm{z}\,\mathrm{tr}_\mathrm{d}(\mathrm{a}) & \text{(Markov property)} \\ \mathrm{tr}_\mathrm{d}(\mathrm{a}\mathrm{t}_{\mathrm{n}+1}^{\mathrm{s}}) & = & \mathrm{x}_\mathrm{s}\mathrm{tr}_\mathrm{d}(\mathrm{a}) & (\mathrm{s} = 1, \ldots, \mathrm{d}-1) \end{array}$$

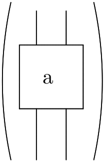
where $\mathrm{a}, \mathrm{b} \in \mathrm{Y}_{\mathrm{d,n}}(\mathrm{q})$

Topological Interpretation of the trace rules

$\mathrm{tr_d}$ 

$= z\,\mathrm{tr_d}$ 

, $\mathrm{tr_d}$ 

$= x_m \mathrm{tr_d}$ 

Link invariants through tr

Isotopy classes of framed links are in bijection with equivalence classes of framed braids under:

Markov Moves

I. Conjugation: $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in \mathcal{F}_{d,n}$

II. Stabilization: $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in \mathcal{F}_{d,n}$

Markov trace

2. Conjugation: $\mathrm{tr_d}(ab) = \mathrm{tr_d}(ba)$
3. Markov Property: $\mathrm{tr_d}(ag_n) = z\,\mathrm{tr_d}(a)$

$a, b \in Y_{d,n}(q)$

The E-system

Problem: The trace $\mathrm{tr_d}$ does not rescale directly.

Answer: The parameters $\mathrm{x_i}$ should be solutions to the following non–linear system of equations, for any $\mathrm{m} \in \mathbb{Z}/\mathrm{d}\mathbb{Z}$:

$$\sum_{\mathrm{s}=0}^{\mathrm{d}-1} \mathrm{x_{m+s} x_{-s}} = \mathrm{x_m} \sum_{\mathrm{s}=0}^{\mathrm{d}-1} \mathrm{x_s x_{-s}} \qquad \text{(E–system)}$$

The full set of solutions of the E-system

Solutions (Gerardín 2012): Parametrised by the non–empty subsets of $\mathbb{Z}/d\mathbb{Z}$.

$$x_D = \frac{1}{|D|} \sum_{s \in D} \exp_s$$

where: D is any non–empty subset of $\mathbb{Z}/d\mathbb{Z}$

$\exp_s(k) := \cos \frac{2\pi sk}{d} + i \sin \frac{2\pi sk}{d}$.

By specializing the trace parameters x_i of tr_d to a solution of the E-system we obtain the specialized trace $tr_{d,D}$ with parameter z.

Constructing invariants for classical links

Let δ now be the following natural homomorphism:

$$\begin{aligned} \delta : \mathbb{C}B_n &\to Y_{d,n}(q) \\ \sigma_i &\mapsto g_i \end{aligned}$$

Let also $\lambda_D := \frac{z-(q-q^{-1})E_D}{z}$, $E_D := \frac{1}{|D|}$, and D is the subset of $\mathbb{Z}/d\mathbb{Z}$ the parametrises a solution of the E-system.

$$\Theta_d(q,\lambda_D)(\widehat{\alpha}) = \left(\frac{1}{z\,\lambda_D}\right)^{n-1} \left(\sqrt{\lambda_D}\right)^{\varepsilon(\alpha)} \mathrm{tr}_{d,D}(\delta(\alpha)),$$

where $\varepsilon(\alpha)$ is the algebraic sum of the exponents of the braiding generators in α.

The subalgebra $Y_{d,n}^{(br)}(q)$

Denote $Y_{d,n}^{(br)}(q) := \delta(\mathbb{C}B_n)$.

- $Y_{d,n}^{(br)}(q)$ is the subalgebra of $Y_{d,n}(q)$ that is generated by $g_1, \ldots, g_{n-1}$.
- The t_i's appear only when the quadratic relation and the inverse relation are applied, and then only in the form of the idempotents e_i.
- One can show that:

$$e_i = \frac{1}{q - q^{-1}}(g_i^3 - g_i)(g_i^2 - 1)$$

- $Y_{d,n}^{(br)}(q)$ coincides with the subalgbera of $Y_{d,n}(q)$ generated by $g_1, \ldots, g_{n-1}, e_1, \ldots, e_{n-1}$.

Theorem (Chlouveraki, Juyumaya, Karvounis, Lambropoulou)

Let $m \in \{1, \ldots, d\}$ and set $E_m := \frac{1}{m}$. Let z be an indeterminate over $\mathbb{C}$. There exists a unique linear Markov trace

$$\mathrm{tr}_{d,m} : \bigcup_{n \geq 0} Y_{d,n}^{(br)}(q) \longrightarrow \mathbb{C}[z]$$

defined inductively on $Y_{d,n}^{(br)}(q)$, for all $n \geq 0$, by the following rules:

$$\begin{array}{lrcll} \text{(i)} & \mathrm{tr}_{d,m}(ab) & = & \mathrm{tr}_{d,m}(ba) & a, b \in Y_{d,n}^{(br)}(q) \\ \text{(ii)} & \mathrm{tr}_{d,m}(1) & = & 1 & \\ \text{(iii)} & \mathrm{tr}_{d,m}(ag_n) & = & z\,\mathrm{tr}_{d,m}(a) & a \in Y_{d,n}^{(br)}(q) \\ \text{(iv)} & \mathrm{tr}_{d,m}(ae_n) & = & E_m\,\mathrm{tr}_{d,m}(a) & a \in Y_{d,n}^{(br)}(q) \\ \text{(v)} & \mathrm{tr}_{d,m}(ae_ng_n) & = & z\,\mathrm{tr}_{d,m}(a) & a \in Y_{d,n}^{(br)}(q), \end{array}$$

For all $a \in \bigcup_{n \geq 0} Y_{d,n}^{(br)}(q)$, we have that $\mathrm{tr}_{d,m}(a) = \mathrm{tr}_{d,D}(a)$ where D is any subset of $\mathbb{Z}/d\mathbb{Z}$ such that $|D| = m$. Note that, in this case $E_m = E_D$.

Comparison to the Homflypt polynomial

Proposition (Chlouveraki, Juyumaya, Karvounis, Lambropoulou)

The invariants Θ_d are topologically equivalent to the Homflypt polynomial on knots.

Theorem (Chlouveraki, Juyumaya, Karvounis, Lambropoulou)

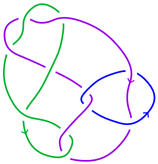
For $d \in \mathbb{Z}_{>1}$, the invariants $\Theta_d(q)$ for classical links are not topologically equivalent to the Homflypt polynomial. Further, the invariants $\Theta_d(q)$ satisfy the skein relation of the Homflypt polynomial on mixed crossings:

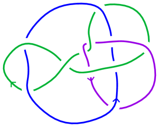
$$\frac{1}{\sqrt{\lambda_D}}\,\Theta_d\left(L_+\right) - \sqrt{\lambda_D}\,\Theta_d\left(L_-\right) = \left(q - q^{-1}\right)\Theta_d\left(L_0\right)$$

Comparison to the Homflypt polynomial

L11n358{0, 1}	L11n418{0, 0}
L11a467{0, 1}	L11a527{0, 0}
L11n325{1, 1}	L11n424{0, 0}
L10n79{1, 1}	L10n95{1, 0}
L11a404{1, 1}	L11a428{0, 1}
L10n76{1, 1}	L11n425{1, 0}

A diagrammatic proof has been completed for the first pair of links (L11n358{0, 1} and L11n418{0, 0}).





The Framization of the Temperley-Lieb algebra

Definition

For $n \geq 3$, the Framization of the Temperley-Lieb algebra $\mathrm{FTL}_{d,n}(q)$ is defined as the quotient of the algebra $\mathrm{Y}_{d,n}(q)$ over the 2-sided ideal generated by the element

$$e_1 e_2 \Big(1 + q(g_1 + g_2) + q^2(g_1 g_2 + g_2 g_1) + q^3 g_1 g_2 g_1 \Big)$$

The representation theory of $\mathrm{FTL}_{d,n}(q)$ has been studied by M. Chlouveraki and G. Pouchin.

Passing of $\mathrm{tr_d}$ to the quotient algebra

Theorem (D.G. Juyumaya, Kontogeorgis, Lambropoulou 2015)

The trace $\mathrm{tr_d}$ passes to $\mathrm{FTL_{d,n}(q)}$ if and only if the parameters of the trace tr satisfy:

$$x_k = -qz\left(\sum_{m\in \mathrm{Sup}_1} \chi_m(k) + (q^2+1)\sum_{m\in \mathrm{Sup}_2} \chi_m(k)\right),$$

$$z = -\frac{1}{q|\mathrm{Sup}_1| + q(q^2+1)|\mathrm{Sup}_2|}.$$

where $\mathrm{Sup}_1 \cup \mathrm{Sup}_2$ (disjoint union) is the support of the Fourier transform of x, χ_m are the characters of $\mathbb{Z}/d\mathbb{Z}$ and x is the complex function on $\mathbb{Z}/d\mathbb{Z}$, that maps 0 to 1 and k to the trace parameter x_k.

Corollary (DG, Kontogeorgis 2015)

In the case where one of the sets Sup_1 or Sup_2 is the empty set we recover all solutions of the E-system.

	x_m	z
$\mathrm{Sup}_1 = \emptyset$	$\sum_{s\in \mathrm{Sup}_2} \exp_s(m)$	$-\frac{q^{-1}E_D}{(q^2+1)}$
$\mathrm{Sup}_2 = \emptyset$	$\sum_{s\in \mathrm{Sup}_1} \exp_s(m)$	$-q^{-1}E_D$

The following is a 1-variable invariant for classical knots and links.

$$\theta_d(q)(\widehat{\alpha}) := \left(-\frac{1+q^2}{qE_D}\right)^{n-1} q^{2\varepsilon(\alpha)} \mathrm{tr}_{d,D}\left(\delta(\alpha)\right) = \Theta_d(q, q^4)(\widehat{\alpha}),$$

Comparison to the Jones polynomial

Proposition (D.G., Juyumaya, Kontogeorgis, Lambropoulou)

The invariants θ_{d} are topologically equivalent to the Jones polynomial on knots.

Theorem (D.G., Juyumaya, Kontogeorgis, Lambropoulou)

For $\mathrm{d} \in \mathbb{Z}_{>1}$, the invariants $\theta_{\mathrm{d}}(\mathrm{q})$ for classical links are not topologically equivalent to the Jones polynomial. Further, the invariants $\theta_{\mathrm{d}}(\mathrm{q})$ satisfy the special skein relation of the Jones polynomial on mixed crossings:

$$\mathrm{q}^{-2}\,\theta_{\mathrm{d}}\,(\mathrm{L}_+) - \mathrm{q}^{2}\,\theta_{\mathrm{d}}\,(\mathrm{L}_-) = (\mathrm{q} - \mathrm{q}^{-1})\,\theta_{\mathrm{d}}\,(\mathrm{L}_0)$$

- All the properties of the invariant Θ_{d} carry through to θ_{d}.
- θ_{d} distinguishes the same pairs of links as Θ_{d}.

Generalizing the invariants

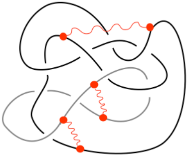
The algebra of braids and ties

The algebra $\mathcal{E}_n(q)$ is the $\mathbb{C}$-algebra that is generated by the elements $b_1, \ldots, b_{n-1}, \epsilon_1, \ldots, \epsilon_{n-1}$ that satisfy the following relations:

$$
\begin{aligned}
b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} & \\
b_i b_j &= b_j b_i & \text{for } |i-j| > 1 \\
\epsilon_i \epsilon_j &= \epsilon_j \epsilon_i & \text{for } |i-j| > 1 \\
\epsilon_i^2 &= \epsilon_i & \\
\epsilon_i b_i &= b_i \epsilon_i & \\
\epsilon_i b_j &= b_j \epsilon_i & \text{for } |i-j| > 1 \\
\epsilon_i \epsilon_j b_i &= b_i \epsilon_i \epsilon_j = \epsilon_j b_i \epsilon_j & \text{for } |i-j| = 1 \\
\epsilon_i b_j b_i &= b_j b_i \epsilon_j & \text{for } |i-j| = 1 \\
b_i^2 &= 1 + (q - q^{-1}) \epsilon_i b_i &
\end{aligned}
$$

The algebra of braids and ties

- Related to Tied Links



- The map $\varphi : \mathcal{E}_n(q) \to Y_{d,n}(q)$ that sends

$$\begin{aligned} b_i &\mapsto g_i \\ \varepsilon_i &\mapsto e_i \end{aligned}$$

is an embedding for $d \geq n$ (Ryom-Hansen and Espinoza 2015).

- For $d \geq n$, $\mathcal{E}_n(q)$ is isomorphic to $Y_{d,n}^{(br)}(q)$.

A Markov trace for $\mathcal{E}_n(q)$

Theorem (Juyumaya, Aicardi)

The algebra $\mathcal{E}_n(q)$ supports a unique Markov trace

$$\rho : \bigcup_{n \geq 0} \mathcal{E}_n(q) \to \mathbb{C}[q^{\pm 1}, z^{\pm 1}, E^{\pm 1}]$$

that can be defined using the following rules:

$$\begin{array}{lrcll}
\text{(i)} & \rho(ab) & = & \rho(ba) & a, b \in \mathcal{E}_n(q) \\
\text{(ii)} & \rho(1_n) & = & 1 & \\
\text{(iii)} & \rho(ab_n) & = & z\rho(a) & a \in \mathcal{E}_n(q) \\
\text{(iv)} & \rho(a\boldsymbol{\varepsilon}_n) & = & E\rho(a) & a \in \mathcal{E}_n(q) \\
\text{(v)} & \rho(a\boldsymbol{\varepsilon}_n b_n) & = & z\rho(a) & a \in \mathcal{E}_n(q),
\end{array}$$

Classical Link invariants from $\mathcal{E}_n(q)$

The Markov trace ρ gives rise to a 3-variable invariant Θ of classical links.

$$\Theta(q, \lambda, E)(\widehat{\alpha}) = \left(\frac{1-\lambda}{\sqrt{\lambda}(q-q^{-1})E} \right)^{n-1} \sqrt{\lambda}^{\varepsilon(\alpha)} \rho(\pi(\alpha)),$$

where $\pi : \mathbb{C}B_n \longrightarrow \mathcal{E}_n(q)$ is the algebra homomorphism that sends $\sigma_i \mapsto b_i$.

For $E = 1/|D|$, the invariant $\Theta(q, \lambda, 1/|D|)$ coincides with the invariant $\Theta_d(q, \lambda_d)$.

A skein relation for Θ

Theorem (Chlouveraki, Juyumaya, Karvounis, Lambropoulou)

Let $\mathcal{L}$ be the set of all oriented links and let q, λ, E be indeterminates. There exists a unique isotopy invariant of classical oriented links

$$\Theta : \mathcal{L} \to \mathbb{C}[q^{\pm 1}, \lambda^{\pm 1}, E^{\pm 1}]$$

defined by the following rules:

1. On crossings involving different components the following skein relation holds:

$$\frac{1}{\sqrt{\lambda}}\,\Theta(L_+) - \sqrt{\lambda}\,\Theta(L_-) = (q - q^{-1})\,\Theta(L_0)$$

   where L_+, L_- and L_0 constitute a Conway triple.

2. For a disjoint union $\mathcal{K} = \sqcup_{i=1}^{r} K_i$ of r knots, with $r > 1$, it holds that:

$$\Theta(\mathcal{K}) = E^{1-r} \prod_{i=1}^{r} P(K_i)$$

   where $P(K_i)$ is the value of the Homflypt polynomial on K_i.

A closed formula for Θ

Theorem (W.B.R. Lickorish)

Let L be an oriented link with n components, then:

$$\Theta(q,\lambda,E)(L) = \sum_{k=1}^{m} \mu^{k-1} E_k \sum_{\pi} \lambda^{\nu(\pi)} P(\pi L),$$

where π are all partitions of the components of L into k (unordered) subsets and $P(\pi L)$ denotes the product of the Homflypt polynomial of the k sublinks of L defined by π. Furthermore, $\nu(\pi)$ is the sum of all linking numbers of pairs of components of L that are distinct sets of π, $E_k = (E^{-1}-1)(E^{-1}-2)\dots(E^{-1}-k+1)$, with $E_1 = 1$ and $\mu = \frac{\lambda^{-1/2}-\lambda^{1/2}}{q-q^{-1}}$

The same result has been proven independently by L. Poulain d'Andecy and E. Wagner.

The partition Temperley-Lieb algebra

Definition

For $n \geq 3$, the partition Temperley-Lieb algebra $PTL_n(q)$ is the quotient of the algebra $\mathcal{E}_n(q)$ over the ideal that is generated by the element:

$$\epsilon_1 \epsilon_2 \left(1 + q(b_1 + b_2) + q^2(b_1 b_2 + b_2 b_1) + q^3 b_1 b_2 b_1\right)$$

Theorem (D.G., Lambropoulou 2016)

The trace ρ on the algebra $\mathcal{E}_n(q)$ passes to the algebra $PTL_n(q)$ if and only if:

$$z = -\frac{q^{-1}E}{q^2 + 1} \quad \text{or} \quad z = -\frac{q^{-1}}{E}.$$

Definition

Let $z = -\frac{q^{-1}E}{q^2+1}$. The following is a 2-variable invariant of classical links:

$$\theta(q, E)(\alpha) := \left(-\frac{q^2 + 1}{q\,E}\right)^{n-1} q^{2\epsilon(\alpha)} \rho(\pi(\alpha)) = \Theta(q, q^4, E)$$

The subalgebra $\mathrm{FTL}_{d,n}^{(br)}(q)$

Define:

$$\mathrm{FTL}_{d,n}^{(br)}(q) = \frac{Y_{d,n}^{(br)}(q)}{\langle e_1 e_2 g_{1,2} \rangle}.$$

Proposition (D.G, Lambropoulou 2016)

For $d \geq n$, the Partition Temperley-Lieb algebra $\mathrm{PTL}_{d,n}(q)$ is isomorphic to the algebra $\mathrm{FTL}_{d,n}^{(br)}(q)$.

Lemma (D.G., Lambropoulou 2016)

For $d \geq n$, the Markov traces $\mathrm{tr}_{d,D}$ on $\mathrm{FTL}_{d,n}^{(br)}(q)$ and ρ on $\mathrm{PTL}_n(q)$ coincide for the case of classical braids.

A skein relation for θ

Theorem (D.G., Lambropoulou 2016)

Let q, E be indeterminates. There exists a unique ambient isotopy invariant of classical oriented links

$$\theta : \mathcal{L} \to \mathbb{C}[\mathrm{q}^{\pm 1}, \mathrm{E}^{\pm 1}]$$

defined by the following rules:

1. On crossings involving different components the following skein relation holds:

$$\mathrm{q}^{-2}\,\theta(\mathrm{L}_+) - \mathrm{q}^{2}\,\theta(\mathrm{L}_-) = (\mathrm{q} - \mathrm{q}^{-1})\,\theta(\mathrm{L}_0)$$

   where L_+, L_- and L_0 constitute a Conway triple.

2. For a disjoint union $\mathcal{K} = \sqcup_{i=1}^{r} \mathrm{K}_i$ of r knots, with $\mathrm{r} > 1$, it holds that:

$$\theta(\mathcal{K}) = \mathrm{E}^{1-r} \prod_{i=1}^{r} \mathrm{V}(\mathrm{K}_i)$$

   where $\mathrm{V}(\mathrm{K}_i)$ is the value of the Jones polynomial on K_i.

A closed formula for θ

Let L be an oriented link with n components. Then:

$$\theta(\mathrm{q}, \mathrm{E})(\mathrm{L}) = \sum_{\mathrm{k}=1}^{\mathrm{m}} (-1)^{\mathrm{k}-1} (\mathrm{q} + \mathrm{q}^{-1})^{\mathrm{k}-1} \mathrm{E}_{\mathrm{k}} \sum_{\pi} \mathrm{q}^{4\nu(\pi)} \mathrm{V}(\pi \mathrm{L}),$$

where the second summation is over all partitions of π of the components of L into k (unordered) subsets and $\mathrm{V}(\pi\mathrm{L})$ denotes the product of the Jones polynomial of the k sublinks of L defined by π. Furthermore, $\nu(\pi)$ is the sum of all linking numbers of pairs of components of L that are distinct sets of π,
$\mathrm{E}_{\mathrm{k}} = (\mathrm{E}^{-1} - 1)(\mathrm{E}^{-1} - 2) \ldots (\mathrm{E}^{-1} - \mathrm{k} + 1)$ and $\mathrm{E}_1 = 1$.

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Thank you very much for your attention!