S-GRAPHS AND THE KASHIWARA $B(\infty)$ CRYSTAL

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Abstract. S-graphs were introduced to study the Kashiwara $B(\infty)$ crystal. They have some remarkable properties which make them of interest in their own right. These are described below together with some hints as to their eventual application to $B(\infty)$.

1. Introduction

Formally the Kashiwara $B(\infty)$ crystal [7] does no more than parameterize a basis of a Verma module. Yet it has a very tight combinatorial structure which results in a parametrization of the simple highest weight modules as well giving a way to determine their tensor product decomposition.

$B(\infty)$ admits a purely combinatorial description using the Littelmann path model [9]. This has the advantage of being valid for any Kac-Moody algebra $\mathfrak{g}$, not just one which is symmetrizable. Remarkably the mysterious Kashiwara functions which determine $B(\infty)$ appear naturally via concatenation of paths.

From the above one may extend Kashiwara duality [8] on $B(\infty)$ to all Kac-Moody algebras [3]. Following this it is natural to introduce "dual Kashiwara" functions. Their interest is that they can determine $B(\infty)$ rather explicitly.

Since the Kashiwara functions are linear it is natural to ask if $B(\infty)$ can be presented as a polyhedral set. This has no known representational interpretation, rather it is a hard question which simply demands an answer. A first attempt was made by Nakashima and Zelevinsky [11] but unfortunately relied on a conjecture which

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almost never holds [10]. Then a beautiful and rather explicit answer to this question was given by Gleizer and Postnikov for $\mathfrak{g}$ simple of type A in [2] using wiring diagrams. They viewed their result as an extension of the Littlewood-Richardson rule for tensor product decomposition. Further to this Berenstein and Zelevinsky [1] established that $B(\infty)$ is polyhedral when $\mathfrak{g}$ is semisimple, equivalently when the Weyl group W is finite. This was achieved using **i**-trails (which we call simply, trails) in the fundamental modules. A disadvantage of this is that trails are not combinatorially defined and almost impossible to determine (at present). These authors also described dual Kashiwara functions [1, Thm. 3.9]; but this used manipulations which are not possible when W is infinite.

The work described here is based on the fact that the dual Kashiwara functions are determined by invariance properties under the action of the Kashiwara operators. This leads to the notion of an S-graph ultimately associated to a simple root α.

A giant S-graph is then proposed which for each simple root α_s must be a disjoint union of S-graphs pertaining to that root, barring one distinguished vertex itself associated to a fixed simple root α_t. Then the vertices of this giant S-graph define the set of dual Kashiwara functions associated to α_t. Conjecturally the set of trails in the fundamental module defined by α_t give functions which form the set of integral points of the convex hull of those obtained from the giant S-graph associated to α_t. So far this has been shown in some special cases and whenever the fundamental module is minuscule.

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2. Definitions

2.1. S**-graphs.** Fix a positive integer n and set $\hat{N} = \{1, 2, \dots, n+1\}$. Let G be a graph whose vertices are labelled by $\hat{N}$, that is to say we have a map from the set $V(G)$ of vertices of G to $\hat{N}$. For all $k \in \hat{N}$, let $V^k(G)$ denote its pre-image in $V(G)$.

The essence of an S-graph is that for each pair $v \in V(G), k \in \hat{N}$, there exists $v' \in V^k(G)$ and an ordered path from v to v'.

Constructing S-graphs should be a question of general interest. However if $|V(G)|$ is finite an obvious difficulty arises if we define an ordered path by aligning arrows on edges. Instead we set $N = \{1, 2, \dots, n\}$ and assign to each edge a non-negative integer $c_k : k \in N$ and require that the c_k increase along an ordered path. This is slightly weaker condition that can be satisfied even if $|V(G)| = 2$.

2.2. **Evaluation.** A key property we require of our S-graphs is that they admit evaluation. This means that to each $v \in V(G)$ we may attach a function z_v such that for all $v \in V^k(G), v' \in V^\ell(G)$ joined by edge with label c_s one has

$$z_v - z_{v'} = c_s(r^k - r^\ell). \tag{1}$$

Here the $r^k : k \in \hat{N}$ are linearly independent functions, ultimately the Kashiwara functions attached to a fixed simple root α_t. The coefficients $c_s : s \in N$ are typically multiplicities of a simple root in a positive root.

2.3. **The Distinguished Element.** An S-graph is required to be connected, so then the set $z_v : v \in V(G)$ is determined by specifying z_v at one vertex v^h.

We specify v^h as follows. Call a pointed chain C a subset $v_{n+1}, v_n, \dots, v_1$ of vertices of G such that $v_i \in V^i(G)$ and v_{i+1}, v_i are joined by the label c_i. An S-graph is required to admit a unique pointed chain and we take v^h to be v_{n+1}. Ultimately z_{v^h} is determined by the way the S-graph fits into the giant S-graph. For the present it can just be taken to be the zero function.

2.4. **Triads.** We impose on an S-graph that vertices with the same label cannot be joined by an edge and that different edges to a given vertex must carry different labels. Then for $n = 1$ there is one connected labelled graph possible and it is indeed an S-graph. For $n = 2$, there are two S-graphs possible. They are chains with four vertices (a, b, c, d) with labels on the edges $(a, b), (c, d)$ coinciding and having a value less than the label on the edge (b, c). In addition the vertices a, d have the same label. We call such a subgraph of an S-graph a "triad". In general we require that the ordering on $c_s; s \in N$ respects the triads.

2.5. **The Canonical S-graphs.** Despite all the tight interlocking conditions we have imposed on S-graphs, there can be several non-isomorphic S-graphs [6, 7.3] for $n \geq 3$ and a given ordering on the set $c_s : s \in N$. Yet for such a choice there is a canonical S-graph determined as a subgraph of a graph $\hat{G}$ whose vertices are equivalences classes of tableaux with $n + 1$ columns satisfying certain boundary conditions. The coefficients $c_s : s \in N$ are inserted into the blocks of the tableaux in a manner which permits the functions $z_v : v \in V(\hat{G})$ to be read off and to be independent of the choice of representative. Moreover $G(\mathbf{c})$ can be read off from $\hat{G}$ just from the structure of the latter as a labelled graph [6, Thm. 7.2].

This combinatorial structure has the interesting (and unexpected) feature that the cardinality of $V(\hat{G})$ is just the Catalan number C_{n+1}, recalling here that $C_n := \frac{1}{n+1}\binom{2n}{n}$. As far as we know $V(\hat{G})$ is a new Catalan set (that is to say a set whose cardinality is a Catalan number). Moreover the graph $\hat{G}$ seems to be quite different to other graphs whose vertices form Catalan sets [6, 7.4.2].

To some extent $\hat{G}$ is irrelevant since the canonical S-sets may be constructed quite independently by what we call binary fusion [4, Sect. 7].

3. Binary Fusion

3.1. Fix a total ordering on $c_s : s \in N$ lifting the natural order and denote this ordered set simply by $\mathbf{c}$. We construct the canonical S-graph $G(\mathbf{c})$ inductively. Here we regard $s \in N$ rather than c_s as the label on edges.

Let c_s be the unique maximal element of $\mathbf{c}$ and set $\mathbf{c}^- := \mathbf{c} \setminus \{c_s\}$ with its induced order. View $N \setminus \{s\}$ (resp. $\hat{N} \setminus \{c_s\}$) as $\{1, 2, \ldots, n-1\}$ (resp. $\{1, 2, \ldots, n\}$) by closing up gaps. Assume that $G(\mathbf{c}^-)$ has been constructed.

Define new graphs G^+, G^- isomorphic to $G(\mathbf{c})$ as unlabelled graphs. Let G^+ be the labelled graph obtained from $G(\mathbf{c}^-)$ by leaving the labels in $[1, s-1]$ unchanged and increasing the labels in $[s, n]$ by 1. Then G^- is defined through the given unlabelled graph isomorphism $\varphi : G^+ \xrightarrow{\sim} G^-$, required to fix all labels with the exception that $\varphi(v) \in V^s(G^-)$ whenever $v \in V^{s+1}(G^+)$.

Define $G(\mathbf{c})$ to be the union of G^+, G^- in which each vertex $v \in V^{s+1}(G^+)$ is joined to $\varphi(v)$ by an edge with label s.

Theorem. *$G(\boldsymbol{c})$ is an S-graph.*

Remarks. This was shown in [4, Sect. 7]. The direct proof of the evaluation property is due to P. Lamprou. In [4, Thm. 8.5], it was shown that $G(\mathbf{c})$ is a subgraph of $\hat{G}$. In [6, Sect. 7] it is shown that $G(\mathbf{c})$ is the unique S-graph of $\hat{G}$ attached to (the totally ordered set) $\mathbf{c}$. In [4, Thm. 8.6] it is shown how the properties of an S-graph can lead to the invariance properties of an S-set required for the construction of dual Kashiwara functions.

3.2. Hypercubes and Simplexes. One may check that $G(\mathbf{c})$ has the structure of a hypercube with some edges missing, indeed exactly those edges which would otherwise join vertices having the same index.

Call $Z(\mathbf{c}) := \{z_v\}_{v \in V(G(\mathbf{c})}$ an S-set. It is said to be of type $t \in I$ if the Kashiwara functions appearing in (1) are of type t - that is to say $r^k = r^k_t$ in the notation of 5.1.

The elements of $Z(\mathbf{c})$ are pairwise distinct if and only if the $c_s : s \in N$ are non-zero and pairwise distinct. In [6, 5.8] it is shown this property may be preserved by collapsing $G(\mathbf{c})$ in a well-defined fashion. Basically triads become triangles. In the most extreme case when the coefficients are non-zero but all equal, all the hypercubes degenerate to the n simplex with edges labelled by the common coefficient. This construction is used to show [6, 5.8] that $Z(\mathbf{c})$ is independent of the lifting of the natural order on $\{c_s\}_{s \in N}$.

The number of canonical S-graphs for each n is smaller than $n!$ because some of the labelled graphs $G(\mathbf{c})$ may be isomorphic. In fact the number of isomorphism classes is the Catalan number C_n, which hence appears for a second time [6, Lemma 6.7].

It is obvious from the binary fusion construction that $|V^i(G(\mathbf{c})| : i \in \hat{N}$ is always a power of 2. On the other hand $|V^i(\hat{G})| = C_{n+1-i}C_i$, in particular $V^n(\hat{G})$ is a Catalan set [6, 3.1].

4. The Convexity Property

There is a convex set associated to the (canonical) S-set $Z(\mathbf{c})$.

Observe that we have two linear order relations on N. The first is the natural order $<$. The second $\prec$ is obtained by a lifting of the natural order on the coefficients $c_s : s \in N$.

Define $s_i : i \in N$ with the property that $s_1 \prec s_2 \prec \ldots \prec s_n$. Relabel $N_k := \{s_i\}_{i=1}^k$ as $\{t_i\}_{i=1}^k$ so that $t_1 < t_2 < \ldots < t_k$. This operation may be viewed as follows. Let s_n be the unique largest element of N with respect to $\prec$. Delete s_n where it appears in N given its natural order and close up the gap, so that N becomes $\{1, 2, \ldots, n-1\}$. Then repeat.

Define $K(\mathbf{c}) = \{c'_k\}_{k \in \mathbb{N}}$ viewed as a subset of $\mathbb{Q}^n$ by

$$0 \leq c'_k \leq c_k, \tag{2}$$

$$c'_{t_{i+1}} - c'_{t_i} \geq \min(0, c_{t_{i+1}} - c_{t_i}). \tag{3}$$

One may check that $K(\mathbf{c})$ is a convex subset of $\mathbb{Q}^n$.

It is shown in [5] that

Theorem. *$Z(\boldsymbol{c})$ is the set of extremal points of $K(\boldsymbol{c})$.*

5. Giant S-graphs

5.1. The Kashiwara $B(\infty)$ Crystal. We follow the construction of the Kashiwara $B(\infty)$ crystal as presented in [3, Sect. 2].

Let I denote the set of labels of the simple roots of $\mathfrak{g}$. Fix a semi-infinite sequence $J = (\ldots, i_j, i_{j-1}, \ldots, i_1)$ with $i_j \in I$ corresponding to successive reduced decompositions of Weyl group elements.

Define B_J to be $\mathbb{N}^{|J|}$, where the j^{th} copy of $\mathbb{N}$ is the elementary crystal (see [3, 2.4]). Then B_J acquires a crystal structure where the Kashiwara operations are described through the Kashiwara functions $r_t^k : t \in I, k \in \mathbb{N}^+$ [3, 2.3.2]. Let b_∞ denote the element of B_J in which all entries are zero and $B_J(\infty)$ the subcrystal of B_J generated by b_∞. It is dependent on J as a subset of B_J viewed as $\mathbb{N}^\infty$, but independent of J as a crystal (result due to Kashiwara in the symmetrizable case and proved in general in [3, 2.5]).

The crystal $B(\infty)$ admits a duality operation $\star$ (result due to Kashiwara in the symmetrizable case [7] and proved in general in [3, 2.5.25]). This allows one to define functions $\varepsilon_t^\star : t \in I$ on $B_J(\infty)$. One seeks to write $\varepsilon_t^\star(b)$ in the form

$$\varepsilon_t^\star(b) = \max_{z \in Z_t} z(b), \forall b \in B_J(\infty), \tag{4}$$

where the $z(b)$ are linear functions on B_J. If this holds then $B_J(\infty)$ must be a polyhedral subset of B_J, that is given by linear inequalities which moreover can be obtained by a simple algorithm involving the functions in the $Z_t : t \in I$.

We regard Z_t as the set of dual Kashiwara functions associated to $t \in I$ eventually eliminating any redundancies so that Z_t is canonically determined by (4).

A basic premise is that the elements of Z_t can be expressed as differences of Kashiwara functions. To this end recall [3, 2.2.2] that one may assign a weight $\mathrm{wt}(b)$ to each element $b \in B_J$, let $\alpha_t^\vee$ denote the coroot defined by $t \in I$ and define the zeroth Kashiwara function r_t^0 by $r_t^0(b) = -\alpha_t^\vee(\mathrm{wt}(b))$, for all $b \in B_J$. Set $z_t^1 = r_t^0 - r_t^1$.

5.2. Giant S-sets.

$(*)$. Fix $t \in I$. A giant S-set of type t is a set Z_t which for all $s \in I \setminus \{t\}$, is a disjoint union of S sets of type s and such that $Z_t \setminus \{z_t^1\}$ is a disjoint union of S-sets of type t.

A giant S-graph (of type t) is the graph whose vertices are Z_t and whose edges are those obtained by the S-graphs defined by the various decompositions of Z_t into S-sets.

Theorem. *If Z_t is a giant S-set of type t, then* (4) *holds.*

5.3. Trails. For small cases one can inductively construct a giant S-set of type t starting from z_t^1 by using the S-sets defined in Section 3 and then verifying property $(*)$. However to show that this works in general is an extremely complex combinatorial problem. One needs in effect a "model" for the functions lying in a giant S-set. All we have at present is the notion of a trail and a dictionary linking trails to linear functions on B_J.

Let ϖ_t be the fundamental weight defined by $t \in I$ and $V(-\varpi_t)$. Let e_s be the simple root vector corresponding to $s \in I$.

A trail K is a sequence of weight vectors $v_j \in V(-\varpi_t) : j \in J$ such that v_{j+1} is a power of e_{i_j} applied to v_j, satisfying certain boundary conditions.

It is not at all easy to determine the trails in $V(-\varpi_t)$. A basic premise is that trails are compatible with adjoining "faces". The latter effect the transition described in Eq. (1). Its proof requires a better understanding of Demazure submodules associated to $V(-\varpi_t)$. Ultimately it truth means that the set of all trails has a combinatorial description.

In the case when $V(-\varpi_t)$ is minuscule, the trails in it are easy to describe and one can construct a giant S-set Z_t of type t. However this does little more than give a proof of [1, Thm. 3.9] albeit one which only involves the elementary combinatorics described here.

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