"Degenerate flags and Schubert varieties"

Martina Lanini
(j/w G. Cerulli Irelli & G. Cerulli Irelli - Littelmann)

University of Edinburgh

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Representation Theory in ΣAMOΣ

Why look at degenerate flag varieties?

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PBW Thm $\Longrightarrow$ filtration on $\mathcal{U}(\mathfrak{n}_-)$:

$$\forall s \in \mathbb{Z}, \qquad \mathcal{U}(\mathfrak{n}_-)_s = \mathrm{Span}_{\mathbb{C}}\{x_1 \cdot \ldots \cdot x_\ell \mid x_i \in \mathfrak{n}_-,\ \ell \leq s\}$$

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<u>Notice</u> $\quad \mathrm{gr}\, \mathcal{U}(\mathfrak{n}_-) = \bigoplus_{s \in \mathbb{Z}} \mathcal{U}(\mathfrak{n}_-)_s / \mathcal{U}(\mathfrak{n}_-)_{s-1} = \mathrm{Sym}(\mathfrak{n}_-) =: S(\mathfrak{n}_-)$

For $\lambda \in P^+ \rightsquigarrow V(\lambda)$ irrep of hw λ

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Explicit description of I_λ only in types A, C, G_2

Feigin-Fourier-Littelmann

Gornitsky

Geometric side of the story: the degenerate flag variety $\mathcal{F}l_\lambda^a$

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  ... and admits a T-stable Whitney stratification

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SCHUBERT VAR'S

VS

DEGENERATE VAR'S

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VS

DEGENERATE FLAGS

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SCHUBERT VAR'S VS DEGENERATE VAR'S

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flag variety

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SCHUBERT VAR'S VS DEGENERATE FLAGS

Type A_ℓ

DEGENERATE FLAGS VS SCHUBERT VAR'S

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SCHUBERT VAR'S VS DEGENERATE FLAGS

$\boxed{\text{Type } A_\ell}$

fund. wts

WLOG: $\lambda = \omega_{i_1} + \omega_{i_2} + \dots + \omega_{i_r}$, $1 \le i_1 < i_2 < \dots < i_r \le \ell$

$$\mathcal{F}l_\lambda \simeq \left\{ W_{i_1} \subset W_{i_2} \subset \dots \subset W_{i_r} \right\} \subset \prod_{j=1}^{r} Gr_{i_j}(\mathbb{C}^{\ell+1})$$

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Expl. $G = SL_4$, $\lambda = \omega_1 + \omega_3$

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Type A_n

(regular λ)

$\{f_1, f_2, \ldots, f_{n+1}\} \subset \mathbb{C}^{n+1}$ basis

$$pr_i : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$$
$$f_k \longmapsto \begin{cases} f_k & k \neq i \\ 0 & k = i \end{cases}$$

Feigin

$$\mathcal{F}l^a_\lambda \simeq \left\{ (V_i) \,\middle|\, pr_i(V_i) \subseteq V_{i+1} \right\} \subset \prod_{i=1}^{n} Gr_i(\mathbb{C}^{n+1})$$

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SCHUBERT VAR'S VS DEGENERATE FLAGS

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Expl. $\boxed{n=2}$ $\{f_1, f_2, f_3\} \subset \mathbb{C}^3$ basis

$$\mathcal{F}l^a_3 \simeq \left\{ (V_1, V_2) \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

SCHUBERT VAR'S VS DEGENERATE FLAGS

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$$\Downarrow$$

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Borel $B \subset$

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SCHUBERT VAR'S VS DEGENERATE FLAGS

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Schubert variety

VS

DEGENERATE FLAGS

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$$\mathcal{F}l^a_\lambda \simeq \left\{ (V_i) \,\middle|\, pr_i(V_i) \subseteq V_{i+1} \right\} \subset \prod_{i=1}^{n} Gr_i(\mathbb{C}^{n+1})$$
$$\| \; \mathcal{F}l^a_{n+1}$$

Expl. $\boxed{n=2}$ $\{f_1, f_2, f_3\} \subset \mathbb{C}^3$ (basis)

$$\mathcal{F}l^a_3$$
$$\text{SI}$$
$$\left\{ (V_1, V_2) \,\middle|\, \begin{array}{l} \dim_{\mathbb{C}} V_1 = 1 \;\; \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$
$$\Psi$$
$$(\langle f_1 \rangle, \langle f_2, f_3 \rangle)$$

SCHUBERT VAR'S VS DEGENERATE FLAGS

SCHUBERT VAR'S VS DEGENERATE FLAGS

$\boxed{\text{Type } A_\ell}$

Schubert varieties are varieties of flags in a given relative position wrt the standard flag $E_\bullet$,

$$E_i = \langle e_1, e_2, \dots e_i \rangle \qquad e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$$

SCHUBERT VAR'S VS DEGENERATE FLAGS

$\boxed{\text{Type } A_n}$

(regular λ)

$$\{f_1, f_2, \dots, f_{n+1}\} \underset{\text{basis}}{\subset} \mathbb{C}^{n+1}$$

$$pr_i : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$$

$$f_k \longmapsto \begin{cases} f_k & k \neq i \\ 0 & k = i \end{cases}$$

Feigin

$$\mathcal{F}l^a_\lambda \simeq \left\{ (V_i) \,\middle|\, pr_i(V_i) \subseteq V_{i+1} \right\} \subset \prod_{i=1}^{n} Gr_i(\mathbb{C}^{n+1})$$

$$\| \quad \mathcal{F}l^a_{n+1}$$

Expl. $\boxed{n=2}$ $\quad \{f_1, f_2, f_3\} \underset{\text{basis}}{\subset} \mathbb{C}^3$

$$\mathcal{F}l^a_3 \cong \left\{ (V_1, V_2) \,\middle|\, \begin{array}{l} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

$$\ni \quad (\langle f_1 \rangle, \langle f_2, f_3 \rangle)$$

SCHUBERT VAR'S vs DEGENERATE FLAGS

Type A_ℓ

Schubert varieties are varieties of flags in a given relative position wrt the standard flag $E_\bullet$,

$$E_i = \langle e_1, e_2, \dots e_i \rangle \qquad e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$$

Expl. $G = SL_4$, $\lambda = \omega_1 + \omega_3$

$$\underbrace{\{e_1, e_2, e_3, e_4\}}_{\text{basis}} \subset \mathbb{C}^4$$

$$\left\{ W_1 \subset W_3 \subset \mathbb{C}^4 \;\middle|\; \begin{array}{c} \dim_{\mathbb{C}} W_i = i \\ W_1 \subset \langle e_1, e_2, e_3 \rangle, \langle e_1 \rangle \subset W_3 \end{array} \right\}$$

$$\overset{\wr\mathrel{|}}{X_{s_2 s_1 s_3}} \subseteq SL_4 / P_{\omega_1 + \omega_3}$$

$$s_2 s_1 s_3 \in \mathfrak{S}_4 / \langle s_2 \rangle \qquad \langle s_2 \rangle = \mathrm{Stab}_{\mathfrak{S}_4}(\omega_1 + \omega_3)$$

DEGENERATE FLAGS vs SCHUBERT VAR'S

Type A_n

(regular λ)

$$\underbrace{\{f_1, f_2, \dots, f_{n+1}\}}_{\text{basis}} \subset \mathbb{C}^{n+1}$$

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Feigin

$$\mathcal{F}l^a_\lambda \simeq \left\{ (V_i) \;\middle|\; pr_i(V_i) \subseteq V_{i+1} \right\} \subset \prod_{i=1}^{n} Gr_i(\mathbb{C}^{n+1})$$

$$\mathcal{F}l^a_\lambda = \mathcal{F}l^a_{n+1}$$

Expl. $\boxed{n=2}$ $\quad \underbrace{\{f_1, f_2, f_3\}}_{\text{basis}} \subset \mathbb{C}^3$

$$\mathcal{F}l^a_3$$

$$\overset{\wr\mathrel{|}}{\left\{ (V_1, V_2) \;\middle|\; \begin{array}{c} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}}$$

$$\begin{array}{cc} \cup\!\!\!\!\mid & \cup\!\!\!\!\mid \\ (\langle f_1 \rangle, \langle f_2, f_3 \rangle) & ([f_1], [f_1, f_2]) \end{array}$$

Thm (Cerulli Irelli – L)

In type A,
degenerate flag varieties are nothing but Schubert varieties!

Expl. $G = SL_4$, $\lambda = \omega_1 + \omega_3$

$\{e_1, e_2, e_3, e_4\} \underset{\text{basis}}{\subset} \mathbb{C}^4$

$$\left\{ W_1 \subset W_3 \subset \mathbb{C}^4 \;\middle|\; \begin{array}{c} \dim_{\mathbb{C}} W_i = i \\ W_1 \subset \langle e_1, e_2, e_3 \rangle, \langle e_1 \rangle \subset W_3 \end{array} \right\}$$

$$\rotatebox{90}{$\subseteq$}$$

$$X_{s_2 s_1 s_3} \subseteq SL_4 / P_{\omega_1 + \omega_3}$$

$s_2 s_1 s_3 \in \mathfrak{S}_4 / \langle s_2 \rangle \longrightarrow = \mathrm{Stab}_{\mathfrak{S}_4}(\omega_1 + \omega_3)$

SCHUBERT VAR'S VS DEGENERATE FLAGS

Expl. $\boxed{n=2}$ $\{f_1, f_2, f_3\} \underset{\text{basis}}{\subset} \mathbb{C}^3$

$$\mathcal{F}l_3^a$$

$$\rotatebox{90}{$\subseteq$}$$

$$\left\{ (V_1, V_2) \;\middle|\; \begin{array}{c} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

$$\rotatebox{90}{$\in$} \qquad\qquad \rotatebox{90}{$\in$}$$

$$(\langle f_1 \rangle, \langle f_2, f_3 \rangle) \qquad ([f_1], [f_1, f_2])$$

$\underline{\text{Thm}}$ (Cerulli Irelli – L)

In type A,
degenerate flag varieties are nothing but Schubert varieties !

More precisely, there is an isomorphism of algebraic varieties :

$$\mathcal{F}l^a_{n+1} \xrightarrow{\sim} X_w \subseteq SL_{2n}/P_{\omega_1+\omega_3+\cdots+\omega_{2n-1}}$$

$$w = s_n s_{n-1} s_{n+1} s_{n-2} s_n s_{n+2} \cdots s_1 s_3 \cdots s_{2n-1}$$

SCHUBERT VAR'S

$\underline{\text{Expl}}$. $G = SL_4$, $\lambda = \omega_1 + \omega_3$

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VS

DEGENERATE FLAGS

$\underline{\text{Expl}}$. $\boxed{n=2}$ $\quad \{f_1, f_2, f_3\} \underset{\text{basis}}{\subset} \mathbb{C}^3$

$$\mathcal{F}l^a_3$$

$$\rotatebox{90}{$\subseteq$}$$

$$\left\{ (V_1, V_2) \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

$$\rotatebox{90}{$\in$} \qquad\qquad \rotatebox{90}{$\in$}$$

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Expl. $G = SL_4$, $\lambda = \omega_1 + \omega_3$

$\underbrace{\{e_1, e_2, e_3, e_4\}}_{\text{basis}} \subset \mathbb{C}^4$

$$\left\{ W_1 \subset W_3 \subset \mathbb{C}^4 \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} W_i = i \\ W_1 \subset \langle e_1, e_2, e_3 \rangle, \langle e_1 \rangle \subset W_3 \end{array} \right\}$$

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$$X_{s_2 s_1 s_3} \subseteq SL_4/P_{\omega_1+\omega_3}$$

$s_2 s_1 s_3 \in \mathfrak{S}_4 / \langle s_2 \rangle \longrightarrow = \mathrm{Stab}_{\mathfrak{S}_4}(\omega_1+\omega_3)$

SCHUBERT VAR'S VS DEGENERATE FLAGS

Expl. $\boxed{n=2}$ $\quad \underbrace{\{f_1, f_2, f_3\}}_{\text{basis}} \subset \mathbb{C}^3$

$$\mathcal{F}l^a_3$$

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$$\left\{ (V_1, V_2) \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

$$\rotatebox{90}{$\in$} \qquad\qquad \rotatebox{90}{$\in$}$$

$$(\langle f_1 \rangle, \langle f_2, f_3 \rangle) \qquad ([f_1], [f_1, f_2])$$

Expl. $\boxed{n=2}$

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$$X_{s_2 s_1 s_3} \subseteq SL_4 / P_{\omega_1+\omega_3}$$

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SCHUBERT VAR'S VS DEGENERATE FLAGS

Expl. $\boxed{n=2}$ $\quad \{f_1, f_2, f_3\} \underset{\text{basis}}{\subset} \mathbb{C}^3$

$$\mathcal{F}l_3^a$$

$$\cong$$

$$\left\{ (V_1, V_2) \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

$$\in$$

$$(\langle f_1\rangle, \langle f_2, f_3\rangle)$$

S

Expl. $\boxed{n=2}$ $\{f_1,f_2,f_3\} \subset \mathbb{C}^3$ basis

$\{e_1,e_2,e_3,e_4\} \subset \mathbb{C}^4$ basis

$$j_1 : \mathbb{C}^3 \longrightarrow \langle e_1,e_2,e_3\rangle \subset \mathbb{C}^4$$
$$f_k \longmapsto e_k$$

$$j_2 : \mathbb{C}^3 \longrightarrow \langle e_1,e_2,e_3,e_4\rangle = \mathbb{C}^4$$
$$f_k \longmapsto \begin{cases} e_k & k \neq 1 \\ e_4 & k = 1 \end{cases}$$

Expl. $G = SL_4$, $\lambda = \omega_1 + \omega_3$

$\{e_1,e_2,e_3,e_4\} \subset \mathbb{C}^4$ basis

$$\left\{ W_1 \subset W_3 \subset \mathbb{C}^4 \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} W_i = i \\ W_1 \subset \langle e_1,e_2,e_3\rangle, \langle e_1\rangle \subset W_3 \end{array} \right\}$$
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$$X_{s_2s_1s_3} \subseteq SL_4/P_{\omega_1+\omega_3}$$

$s_2s_1s_3 \in \mathfrak{S}_4/\langle s_2\rangle$, $\langle s_2\rangle = \mathrm{Stab}_{\mathfrak{S}_4}(\omega_1+\omega_3)$

SCHUBERT VAR'S VS DEGENERATE FLAGS

S

Expl. $\boxed{n=2}$ $\{f_1,f_2,f_3\} \subset \mathbb{C}^3$ basis

$$\mathcal{F}l_3^a$$
$$\cong$$
$$\left\{ (V_1,V_2) \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

$\in$ $(\langle f_1\rangle, \langle f_2,f_3\rangle)$ $\quad$ $\in$ [illegible]

Expl. $\boxed{n=2}$ $\quad \underbrace{\{f_1, f_2, f_3\}}_{\text{basis}} \subset \mathbb{C}^3$

$$\underbrace{\{e_1, e_2, e_3, e_4\}}_{\text{basis}} \subset \mathbb{C}^4$$

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$$f_k \longmapsto e_k$$

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$$f_k \longmapsto \begin{cases} e_k & k \neq 1 \\ e_4 & k = 1 \end{cases}$$

$$\mathcal{F}l^a_3 \longrightarrow Gr_1(\mathbb{C}^4) \times Gr_3(\mathbb{C}^4)$$
$$(V_1, V_2) \longmapsto \left(j_1(V_1),\ j_2(V_2) \oplus \mathbb{C}e_1 \right)$$

Expl. $G = SL_4$, $\lambda = \omega_1 + \omega_3$

$$\underbrace{\{e_1, e_2, e_3, e_4\}}_{\text{basis}} \subset \mathbb{C}^4$$

$$\left\{ W_1 \subset W_3 \subset \mathbb{C}^4 \;\middle|\; \begin{array}{c} \dim_{\mathbb{C}} W_i = i \\ W_1 \subset \langle e_1, e_2, e_3 \rangle, \langle e_1 \rangle \subset W_3 \end{array} \right\}$$
$$\cong$$
$$X_{s_2 s_1 s_3} \subseteq SL_4 / P_{\omega_1 + \omega_3}$$

$$s_2 s_1 s_3 \in \mathfrak{S}_4 / \langle s_2 \rangle \longrightarrow = \mathrm{Stab}_{\mathfrak{S}_4}(\omega_1 + \omega_3)$$

SCHUBERT VAR'S VS DEGENERATE FLAGS

S

Expl. $\boxed{n=2}$ $\quad \underbrace{\{f_1, f_2, f_3\}}_{\text{basis}} \subset \mathbb{C}^3$

$$\mathcal{F}l^a_3$$
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$$\left\{ (V_1, V_2) \;\middle|\; \begin{array}{c} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

$$\ni \qquad\qquad \ni$$
$$(\langle f_1 \rangle, \langle f_2, f_3 \rangle) \qquad (\langle f_1 \rangle, \langle f_1, f_2 \rangle)$$

$\boxed{\text{Type } C_{2r}}$

SCHUBERT VAR'S VS DEGENERATE FLAGS

SCHUBERT VAR'S VS DEGENERATE FLAGS

Type C_r

SCHUBERT VAR'S VS DEGENERATE FLAGS

$\{e_1, \dots, e_{2r}\} \subset \mathbb{C}^{2r}$ basis

$(\; , \;) : \mathbb{C}^{2r} \times \mathbb{C}^{2r} \longrightarrow \mathbb{C}$

given by $\begin{array}{cc} \overbrace{}^{r} \; \overbrace{}^{r} & \\ \left(\begin{array}{c|c} 0 & \begin{smallmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & -1 \\ & \cdot^{\cdot^{\cdot}} & \\ -1 & & \end{smallmatrix} & 0 \end{array} \right) & \begin{array}{l} \}\, r \\ \}\, r \end{array} \end{array}$

SCHUBERT VAR'S VS DEGENERATE FLAGS

SCHUBERT VAR'S vs DEGENERATE FLAGS

SCHUBERT VAR'S vs DEGENERATE FLAGS

Type C_r

$\underbrace{\{e_1,\dots,e_{2r}\}}_{\text{basis}} \subset \mathbb{C}^{2r}$

$(\ ,\ ):\mathbb{C}^{2r}\times\mathbb{C}^{2r}\longrightarrow\mathbb{C}$

given by

$$\begin{array}{cc} \begin{array}{cc} \overbrace{}^{r} & \overbrace{}^{r} \end{array} & \\ \left(\begin{array}{c|c} 0 & \begin{matrix} & & 1 \\ & ⋰ & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & -1 \\ & ⋰ & \\ -1 & & \end{matrix} & 0 \end{array}\right) & \begin{array}{l} \}\,r \\ \}\,r \end{array} \end{array}$$

$\mu=\sum a_i\omega_i \,:\, a_i=a_{2r-i}$ $\quad$ (e.g. $\mu=\omega_1+\omega_3+\dots+\omega_{2r-3}+\omega_{2r-1}$)

"symplectic"

$$\begin{array}{ccc} (W_{i_j}) & \longmapsto & (W^{\perp}_{2r-i_j}) \\ \cap & & \cap \\ \mathcal{F}l_\mu(sl_{2r}) & \xrightarrow{\ \tau\ } & \mathcal{F}l_\mu(sl_{2r}) \end{array}$$

SCHUBERT VAR'S VS DEGENERATE FLAGS

$\boxed{\text{Type } C_r}$

$\{e_1, \dots, e_{2r}\} \subset \mathbb{C}^{2r}$ basis

$(\, , \,) : \mathbb{C}^{2r} \times \mathbb{C}^{2r} \longrightarrow \mathbb{C}$

given by $\begin{array}{cc} \begin{array}{cc} \overbrace{\quad}^{r} & \overbrace{\quad}^{r} \end{array} & \\ \begin{pmatrix} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} \\ \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & 0 \end{pmatrix} & \begin{array}{l} \}\, r \\ \}\, r \end{array} \end{array}$

$\mathcal{F}l_\mu(sp_{2r}) \simeq \mathcal{F}l_\mu(sl_{2r})^\tau$

$\mu = \sum a_i \omega_i \; : \; a_i = a_{2r-i}$ "symplectic" $\left(\text{e.g. } \mu = \omega_1 + \omega_3 + \dots + \omega_{2r-3} + \omega_{2r-1} \right)$

$$\begin{array}{ccc} (W_{i_j}) & \longmapsto & (W^{\perp}_{2r-i_j}) \\ \cap & & \cap \\ \mathcal{F}l_\mu(sl_{2r}) & \xrightarrow{\tau} & \mathcal{F}l_\mu(sl_{2r}) \end{array}$$

SCHUBERT VAR'S VS DEGENERATE FLAGS

SCHUBERT VAR'S

Type C_r

$$\mathcal{F}l_\mu(sp_{2r}) \simeq \mathcal{F}l_\mu(sl_{2r})^{\tau}$$

$\underbrace{\{e_1,\dots,e_{2r}\}}_{\text{basis}} \subset \mathbb{C}^{2r}$

$(\ ,\ ) : \mathbb{C}^{2r} \times \mathbb{C}^{2r} \longrightarrow \mathbb{C}$

given by
$$\begin{array}{c c}
\begin{array}{cc} \overbrace{\quad}^{r} & \overbrace{\quad}^{r} \end{array} & \\
\left(\begin{array}{c|c} 0 & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & -1 \\ & \iddots & \\ -1 & & \end{matrix} & 0 \end{array}\right) & \begin{array}{l} \}\, r \\ \}\, r \end{array}
\end{array}$$

$\mu = \sum a_i \omega_i \ : \ a_i = a_{2r-i}$ ("symplectic") $(e.g.\ \mu = \omega_1 + \omega_3 + \dots + \omega_{2r-3} + \omega_{2r-1})$

$$(W_i) \longmapsto (W_{2r-i}^{\perp})$$

$$\mathcal{F}l_\mu(sl_{2r}) \xrightarrow{\ \tau\ } \mathcal{F}l_\mu(sl_{2r})$$

SCHUBERT VAR'S

VS

DEGENERATE FLAGS

Type C_m

(regular λ)

$\underbrace{\{f_1, f_2, \dots, f_{2m}\}}_{\text{basis}} \subset \mathbb{C}^{2m}$

$(\ ,\ )^a : \mathbb{C}^{2m} \times \mathbb{C}^{2m} \longrightarrow \mathbb{C}$

given by
$$\begin{array}{c c}
\begin{array}{ccc} \overbrace{\quad}^{m-1} & \overbrace{\quad}^{m-1} & \overbrace{\quad}^{2} \end{array} & \\
\left(\begin{array}{c|c|c} 0 & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} & 0 \\ \hline \begin{matrix} & & -1 \\ & \iddots & \\ -1 & & \end{matrix} & 0 & 0 \\ \hline 0 & 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right) & \begin{array}{l} \}\, m-1 \\ \}\, m-1 \\ \}\, 2 \end{array}
\end{array}$$

DEGENERATE FLAGS

SCHUBERT VAR'S VS DEGENERATE FLAGS

Type C_r

$\{e_1,\dots,e_{2r}\} \subset \mathbb{C}^{2r}$ basis

$(\ , \ ) : \mathbb{C}^{2r} \times \mathbb{C}^{2r} \longrightarrow \mathbb{C}$

given by
$$\begin{array}{c} \begin{array}{cc} \overbrace{\qquad}^{r} & \overbrace{\qquad}^{r} \end{array} \\ \left(\begin{array}{c|c} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & 0 \end{array}\right) \begin{array}{l} \}\, r \\ \}\, r \end{array} \end{array}$$

$$\mathcal{F}l_\mu(sp_{2r}) \simeq \mathcal{F}l_\mu(sl_{2r})^{\tau}$$

$\underbrace{\mu}_{\text{"symplectic"}} = \sum a_i \omega_i \ : \ a_i = a_{2r-i}$ (e.g. $\mu = \omega_1 + \omega_3 + \dots + \omega_{2r-3} + \omega_{2r-1}$)

$$(W_{i_j}) \longmapsto (W^{\perp}_{2r-i_j})$$

$$\mathcal{F}l_\mu(sl_{2r}) \xrightarrow{\ \tau\ } \mathcal{F}l_\mu(sl_{2r})$$

Type C_m

(regular λ)

$\{f_1, f_2, \dots, f_{2m}\} \subset \mathbb{C}^{2m}$ basis

$(\ , \ )^a : \mathbb{C}^{2m} \times \mathbb{C}^{2m} \longrightarrow \mathbb{C}$

given by
$$\begin{array}{c} \begin{array}{ccc} \overbrace{\qquad}^{m-1} & \overbrace{\qquad}^{m-1} & \overbrace{\quad}^{2} \end{array} \\ \left(\begin{array}{c|c|c} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} & 0 \\ \hline \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & 0 & 0 \\ \hline 0 & 0 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{array}\right) \begin{array}{l} \}\, m-1 \\ \}\, m-1 \\ \}\, 2 \end{array} \end{array}$$

$$\mathcal{F}l^a_\lambda(sl_{2m}) \xrightarrow{\ \tau^a\ } \mathcal{F}l^a_\lambda(sl_{2m})$$

$$(V_{i_j}) \longmapsto (V^{\perp^a}_{i_j^*})$$

λ symplectic (e.g. $\lambda = \omega_1 + \omega_2 + \omega_3 + \dots + \omega_{2m-2} + \omega_{2m-1}$)

SCHUBERT VAR'S VS DEGENERATE FLAGS

Type C_r

$\{e_1,\dots,e_{2r}\} \underset{\text{basis}}{\subset} \mathbb{C}^{2r}$

$(\ ,\ ): \mathbb{C}^{2r} \times \mathbb{C}^{2r} \longrightarrow \mathbb{C}$

given by
$$\begin{array}{cc|cc}
\begin{array}{cc} r & r \end{array} & \\
\left(\begin{array}{c|c} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & 0 \end{array}\right) & \begin{array}{l} \}\, r \\ \}\, r \end{array}
\end{array}$$

$$\mathcal{F}l_\mu(sp_{2r}) \simeq \mathcal{F}l_\mu(sl_{2r})^{\tau}$$

$\underset{\text{"symplectic"}}{\mu} = \sum a_i \omega_i \ :\ a_i = a_{2r-i}$ (e.g. $\mu = \omega_1 + \omega_3 + \dots + \omega_{2r-3} + \omega_{2r-1}$)

$$\begin{array}{ccc}
(W_{i_j}) & \longmapsto & (W^{\perp}_{2r-i_j}) \\
\cap & & \cap \\
\mathcal{F}l_\mu(sl_{2r}) & \xrightarrow{\ \tau\ } & \mathcal{F}l_\mu(sl_{2r})
\end{array}$$

SCHUBERT VAR'S VS DEGENERATE FLAGS

(regular λ)

$\{f_1, f_2, \dots, f_{2m}\} \underset{\text{basis}}{\subset} \mathbb{C}^{2m}$

$(\ ,\ )^a: \mathbb{C}^{2m} \times \mathbb{C}^{2m} \longrightarrow \mathbb{C}$

given by
$$\begin{array}{cc}
\begin{array}{ccc} m-1 & m-1 & 2 \end{array} & \\
\left(\begin{array}{c|c|c} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} & 0 \\ \hline \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & 0 & 0 \\ \hline 0 & 0 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{array}\right) & \begin{array}{l} \}\, m-1 \\ \}\, m-1 \\ \}\, 2 \end{array}
\end{array}$$

Type C_m

$$\begin{array}{ccc}
\mathcal{F}l^a_\lambda(sl_{2m}) & \xrightarrow{\ \tau^a\ } & \mathcal{F}l^a_\lambda(sl_{2m}) \\
\Cup & & \Cup \\
(V_{i_j}) & \longmapsto & (V^{\perp^a}_{i_j^*})
\end{array}$$

λ symplectic (e.g. $\lambda = \omega_1 + \omega_2 + \omega_3 + \dots + \omega_{2m-2} + \omega_{2m-1}$)

$$\mathcal{F}l^a_\lambda(sp_{2m}) \overset{[\text{FFL}]}{\simeq} \mathcal{F}l^a_\lambda(sl_{2m})^{\tau^a}$$

SCHUBERT VAR'S vs DEGENERATE FLAGS

$\boxed{\text{Type } C_r}$

$\{e_1, \dots, e_{2r}\} \subset \mathbb{C}^{2r}$ basis

$(\ , \ ) : \mathbb{C}^{2r} \times \mathbb{C}^{2r} \longrightarrow \mathbb{C}$

given by $$\begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{r} & \overbrace{\quad}^{r} \end{array} \\ \left(\begin{array}{c|c} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & 0 \end{array}\right) \begin{array}{l} \}\, r \\ \}\, r \end{array} \end{array}$$

$\mathcal{F}l_\mu(sp_{2r}) \simeq \mathcal{F}l_\mu(sl_{2r})^\tau$

$\mu = \sum a_i \omega_i \ : \ a_i = a_{2r-i}$ "symplectic" (e.g. $\mu = \omega_1 + \omega_3 + \dots + \omega_{2m-5} + \omega_{2m-3}$)

$(W_{i_j}) \longmapsto (W^{\perp}_{2r-i_j})$

$\mathcal{F}l_\mu(sl_{2m-2}) \xrightarrow{\tau} \mathcal{F}l_\mu(sl_{2m-2})$

$\boxed{\text{Type } C_m}$

(regular λ)

$\{f_1, f_2, \dots, f_{2m}\} \subset \mathbb{C}^{2m}$ basis

$(\ , \ )^a : \mathbb{C}^{2m} \times \mathbb{C}^{2m} \longrightarrow \mathbb{C}$

given by $$\begin{array}{c} \begin{array}{ccc} \overbrace{\quad}^{m-1} & \overbrace{\quad}^{m-1} & \overbrace{\quad}^{2} \end{array} \\ \left(\begin{array}{c|c|c} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} & 0 \\ \hline \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & 0 & 0 \\ \hline 0 & 0 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{array}\right) \begin{array}{l} \}\, m-1 \\ \}\, m-1 \\ \}\, 2 \end{array} \end{array}$$

$\mathcal{F}l^a_\lambda(sl_{2m}) \xrightarrow{\tau^a} \mathcal{F}l^a_\lambda(sl_{2m})$

$(V_{i_j}) \longmapsto (V^{\perp^a}_{i_j^*})$

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$\mathcal{F}l^a_\lambda(sp_{2m}) \underset{\text{[FFL]}}{\simeq} \mathcal{F}l^a_\lambda(sl_{2m})^{\tau^a}$

SCHUBERT VAR'S vs DEGENERATE FLAGS

Thm (Cerulli Irelli – L)

In type A,
degenerate flag varieties are nothing but Schubert varieties!

More precisely, there is an isomorphism of algebraic varieties:

$$\mathcal{F}l^a_{n+1} \xrightarrow{\sim} X_w \subseteq SL_{2n}/P_{\omega_1+\omega_3+\dots+\omega_{2n-1}}$$

$W = s_n s_{n-1} s_{n+1} s_{n-2} s_n s_{n+2} \cdots s_1 s_3 \cdots s_{2n-1}$

Expl. $G = SL_4$, $\lambda = \omega_1 + \omega_3$

$\{e_1, e_2, e_3, e_4\} \underset{\text{basis}}{\subset} \mathbb{C}^4$

$$\left\{ W_1 \subset W_3 \subset \mathbb{C}^4 \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} W_i = i \\ W_1 \subset \langle e_1, e_2, e_3\rangle, \langle e_1\rangle \subset W_3 \end{array} \right\}$$

$$\text{SI}$$

$$X_{s_2 s_1 s_3} \subseteq SL_4/P_{\omega_1+\omega_3}$$

$s_2 s_1 s_3 \in \mathfrak{S}_4 / \langle s_2 \rangle \longrightarrow = \mathrm{Stab}_{\mathfrak{S}_4}(\omega_1+\omega_3)$

SCHUBERT VAR'S vs DEGENERATE FLAGS

Expl. $\boxed{n=2}$ $\quad \{f_1, f_2, f_3\} \underset{\text{basis}}{\subset} \mathbb{C}^3$

$$\mathcal{F}l^a_3$$

$$\text{SI}$$

$$\left\{ (V_1, V_2) \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

$$\begin{array}{cc} \Psi & \Psi \\ (\langle f_1\rangle, \langle f_2, f_3\rangle) & (\langle f_1\rangle, \langle f_1, f_2\rangle) \end{array}$$

S

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ζ

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Expl. $G = SL_4$, $\lambda = \omega_1+\omega_3$

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SCHUBERT VAR'S VS DEGENERATE FLAGS

$\mathcal{S}$

Expl. $\boxed{n=2}$ $\{f_1, f_2, f_3\} \underset{\text{basis}}{\subset} \mathbb{C}^3$

$$\mathcal{F}l^a_3$$

$$\mathrel{\rotatebox{90}{$\subseteq$}} \left\{ (V_1, V_2) \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}} V_1 = 1 \quad \dim_{\mathbb{C}} V_2 = 2 \\ pr_1(V_1) \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}$$

$\ni$ $(\langle f_1\rangle, \langle f_2, f_3\rangle)$ $\qquad \ni$ $([f_1], [f_1, f_2])$

DEGENERATE FLAGS VS SCHUBERT VAR'S

Type C_{2r}

$$\mathcal{F}l_{\mu}(sp_{2r}) \simeq \mathcal{F}l_{\mu}(sl_{2r})^{\tau}$$

$\mu = \sum a_i \omega_i \ : \ a_i = a_{2r-i}$ $\left(\text{e.g. } \mu = \omega_1 + \omega_3 + \dots + \omega_{2m-5} + \omega_{2m-3}\right)$

"symplectic"

$$\begin{tikzcd}
(W_{i_j}) \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & (W^{\perp}_{2r-i_j}) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\mathcal{F}l_{\mu}(sl_{2m-2}) \arrow[r, "\tau"] & \mathcal{F}l_{\mu}(sl_{2m-2}) \\
\mathcal{F}l^{a}_{\lambda}(sl_{2m}) \arrow[r, "\tau^{a}"] \arrow[u, hook, "\zeta"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathcal{F}l^{a}_{\lambda}(sl_{2m}) \arrow[u, hook, "\zeta"'] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
(V_{i_j}) \arrow[r, mapsto] & (V^{\perp^{a}}_{i_j^{*}})
\end{tikzcd}$$

Type C_{2m}

λ symplectic $\left(\text{e.g. } \lambda = \omega_1 + \omega_2 + \omega_3 + \dots + \omega_{2m-2} + \omega_{2m-1}\right)$

[FFL]

$$\mathcal{F}l^{a}_{\lambda}(sp_{2m}) \simeq \mathcal{F}l^{a}_{\lambda}(sl_{2m})^{\tau^{a}}$$

SCHUBERT VAR'S VS DEGENERATE FLAGS

DEGENERATE FLAGS **VS** SCHUBERT VAR'S

Type C_{2r} $\qquad \mathcal{F}l_{\mu}(sp_{2r}) \simeq \mathcal{F}l_{\mu}(sl_{2r})^{\tau}$

$\mu = \sum a_i \omega_i \ : \ a_i = a_{2r-i}$ (e.g. $\mu = \omega_1 + \omega_3 + \dots + \omega_{2m-5} + \omega_{2m-3}$)

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\mathcal{F}l_{\mu}(sl_{2m-2}) \arrow[r, "\tau"] & \mathcal{F}l_{\mu}(sl_{2m-2}) \\
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\end{tikzcd}$$

Type C_{2m}

$$(V_{i_j}) \longmapsto (V^{\perp^{a}}_{i_j^{*}})$$

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[FFL]

$$\mathcal{F}l^{a}_{\lambda}(sp_{2m}) \simeq \mathcal{F}l^{a}_{\lambda}(sl_{2m})^{\tau^{a}}$$

SCHUBERT VAR'S **VS** DEGENERATE FLAGS

DEGENERATE FLAGS **VS** SCHUBERT VAR'S

<u>Thm</u> (Cerulli Irelli-L)

Also in type C, degenerate flag varieties are Schubert varieties!

$\boxed{\text{Type } C_{2r}}$

$$\mathcal{F}l_\mu(sp_{2r}) \simeq \mathcal{F}l_\mu(sl_{2r})^\tau$$

$\mu = \sum a_i \omega_i \; : \; a_i = a_{2r-i}$ (e.g. $\mu = \omega_1 + \omega_3 + \dots + \omega_{2m-5} + \omega_{2m-3}$)

"symplectic"

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$$\begin{tikzcd}
\mathcal{F}l_\mu(sl_{2m-2}) \arrow[r, "\tau"] & \mathcal{F}l_\mu(sl_{2m-2}) \\
\mathcal{F}l^a_\lambda(sl_{2m}) \arrow[u, "\zeta"] \arrow[r, "\tau^a"] & \mathcal{F}l^a_\lambda(sl_{2m}) \arrow[u, "\zeta"']
\end{tikzcd}$$

$\boxed{\text{Type } C_{2m}}$

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$$\mathcal{F}l^a_\lambda(sp_{2m}) \overset{[FFL]}{\simeq} \mathcal{F}l^a_\lambda(sl_{2m})^{\tau^a}$$

SCHUBERT VAR'S **VS** DEGENERATE FLAGS

Remarks.

Remarks. ⓪ Everything works for singular λ too.

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① Classical results on Schubert varieties (Mehta-Ramanathan, Ramanan-Ramanathan, ...)

$\Downarrow$

Thm (Feigin-Finkelberg,) In type A , $\mathcal{F}l_\lambda^a$ is projectively normal, with rational singularities.

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Thm (Feigin-Finkelberg,-Littelmann) In type A & C, $\mathcal{F}l_\lambda^a$ is projectively normal, with rational singularities.

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$$\mathcal{F}l^a_{n+1} \xrightarrow{\sim} X_w \subset SL_{2n}/P_{\omega_1+\omega_3+\ldots+\omega_{2n-1}}$$ T-eqvt

$$\rightsquigarrow H^\bullet_T(\mathcal{F}l^a_{n+1}) \simeq H^\bullet_T(X_w)$$

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$$\mathcal{F}l^a_{n+1} \xrightarrow{\sim} X_w \subset SL_{2n}/P_{\omega_1+\omega_3+\dots+\omega_{2n-1}}$$

T-eqvt

$$\rightsquigarrow H^\bullet_T(\mathcal{F}l^a_{n+1}) \simeq H^\bullet_T(X_w)$$

Their T-stable stratification coincides with the one given by the orbits of the Borel $\subset SL_{2n}$, Borel $= \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$

$$\rightsquigarrow IH^\bullet_T(\mathcal{F}l^a_{n+1}) \simeq IH^\bullet_T(X_w)$$

③ Thm (Cerulli Irelli-L) Feigin-Finkelberg desingularisations are Bott-Samelson varieties

③ $\underline{\text{Thm}}$ (Cerulli Irelli-L) Feigin-Finkelberg desingularisations are Bott-Samelson varieties

$\underline{\text{Expl}}$. $\boxed{n=2}$

$$FF = \left\{ \left(Z_{(1,1)}, Z_{(1,2)}, Z_{(2,2)}\right) \;\middle|\; \begin{array}{l} Z_{(1,1)} \in Gr_1(\langle f_1, f_2, f_3\rangle) \\ pr_1 Z_{(1,1)} \subseteq Z_{(1,2)} \in Gr_1(\langle f_2, f_3\rangle) \\ Z_{(1,2)} \subseteq Z_{(2,2)} \in Gr_1(\langle f_1, f_2, f_3\rangle) \end{array} \right\} \longrightarrow \mathcal{F}l_3^a$$

$$\left(Z_{(1,1)}, Z_{(1,2)}, Z_{(2,2)}\right) \longmapsto \left(Z_{(1,1)}, Z_{(2,2)}\right) \in \mathcal{F}l_3^a$$

$$\left(Z_{(1,1)}, Z_{(1,2)}, Z_{(2,2)}\right) \longmapsto \left(\begin{pmatrix} \langle e_1\rangle \\ \cap \\ \langle e_1, e_2\rangle \\ \cap \\ \langle e_1, e_2, e_3\rangle \end{pmatrix}, \begin{pmatrix} \langle e_1\rangle \\ \cap \\ j_1(Z_{(1,2)}) \oplus \langle e_1\rangle \\ \cap \\ \langle e_1, e_2, e_3\rangle \end{pmatrix}, \begin{pmatrix} j_1(Z_{(1,1)}) \\ \cap \\ j_1(Z_{(1,2)}) \oplus \langle e_1\rangle \\ \cap \\ \langle e_1, e_2, e_3\rangle \end{pmatrix}, \begin{pmatrix} j_1(Z_{(1,1)}) \\ \cap \\ j_1(Z_{(1,2)}) \oplus \langle e_1\rangle \\ \cap \\ j_2(Z_{(2,2)}) \oplus \langle e_1\rangle \end{pmatrix} \right) \longmapsto \left(j_1(Z_{(1,1)}), j_2(Z_{(2,2)}) \oplus \langle e_1\rangle\right)$$

$$BS_{\underline{s_2 s_1 s_3}} = \left\{ \left(\begin{pmatrix} \langle e_1\rangle \\ \cap \\ \langle e_1, e_2\rangle \\ \cap \\ \langle e_1, e_2, e_3\rangle \end{pmatrix}, \begin{pmatrix} \langle e_1\rangle \\ \cap \\ W_2^{(1)} \\ \cap \\ \langle e_1, e_2, e_3\rangle \end{pmatrix}, \begin{pmatrix} W_1^{(2)} \\ \cap \\ W_2^{(1)} \\ \cap \\ \langle e_1, e_2, e_3\rangle \end{pmatrix}, \begin{pmatrix} W_1^{(2)} \\ \cap \\ W_2^{(1)} \\ \cap \\ W_3^{(3)} \end{pmatrix} \right) \;\middle|\; \dim W_i^{(k)} = i \right\} \longrightarrow X_{s_2 s_1 s_3}$$

$$\left(\ldots, \begin{pmatrix} W_1^{(2)} \\ \cap \\ W_2^{(1)} \\ \cap \\ W_3^{(3)} \end{pmatrix} \right) \longmapsto \left(W_1^{(2)}, W_3^{(3)}\right) \in X_{s_2 s_1 s_3}$$

What about the original, algebraic question? $\left(V(\lambda)^a \simeq S(\mathfrak{n}_-)/I_\lambda\right) \to ?$

What about the original, algebraic question? $\left(V(\lambda)^a \simeq S(\mathfrak{n}_-)/ I_\lambda \right)$?

Let $\underset{\text{subvect-sp}}{U \subseteq} \mathrm{Mat}_{n\times n}(\mathbb{C})$,

$$U \underset{\text{v. sp.}}{\cong} \left\{ \left(\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array} \right) \middle| A \in U \right\} \subseteq \mathfrak{sl}_{2n}(\mathbb{C})$$

$$M, N \in \Longrightarrow [M,N] := MN - NM = 0$$

What about the original, algebraic question? $\left(V(\lambda)^a \simeq S(n_-) / I_\lambda \right) \to ?$

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$$M, N \in \Rightarrow [M,N] := MN - NM = 0$$

$$n_- \in \mathfrak{sl}_n(\mathbb{C}) \rightsquigarrow n_- \underset{\text{v.sp.}}{\cong} \left\{ \left(\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array} \right) \middle| A \in n_- \right\} \underset{\text{L.alg.}}{\cong} n_-^a \rightsquigarrow \mathcal{U}(n_-^a) = S(n_-^a) = S(n_-)$$

What about the original, algebraic question? $\left(V(\lambda)^a \simeq S(n_-)/I_\lambda \right)$?

Let $U \subseteq \mathrm{Mat}_{n\times n}(\mathbb{C})$, subvect-sp

$$U \underset{\text{v. sp.}}{\cong} \left\{ \left(\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array} \right) \middle| A \in U \right\} \subseteq \mathfrak{sl}_{2n}(\mathbb{C})$$

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$$\cap$$

$$\left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} = \tilde{b} \subset \mathfrak{sl}_{2n}(\mathbb{C})$$

$\rightsquigarrow$ any $\mathcal{U}(\tilde{b})$-module has naturally a structure of $S(n_-^a) = S(n_-)$-module

Let $\mu \in \widetilde{P}^+ \rightsquigarrow \widetilde{V}(\mu)$ irrep of hw μ

$\mathfrak{sl}_{2n}$

$\mathfrak{sl}_{2n}$

Let $\mu \in \tilde{P}^+ \rightsquigarrow \tilde{V}(\mu)$ irrep of hw μ

$$\tilde{V}(\mu) = \bigoplus \tilde{V}(\mu)_\nu \qquad \tilde{V}(\mu)_{x\mu} \cong \mathbb{C} \cdot v_{x\mu} \quad x \in \tilde{W} = \tilde{\mathfrak{S}}_{2n}$$

Let $\mu \in \tilde{P}^+ \leadsto \tilde{V}(\mu)$ irrep of hw μ

$\mathfrak{sl}_{2n}$

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The Demazure module $\tilde{V}(\mu)_x$ is the $\mathcal{U}(\tilde{\mathfrak{b}})$-submodule of $\tilde{V}(\mu)$ generated by $v_{x\mu}$

$$\tilde{V}(\mu)_x \supseteq \mathcal{U}(\tilde{\mathfrak{b}}) . v_{x\mu}$$

$\mathfrak{sl}_{2n}$

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<u>Thm</u> (Cerulli Irelli - L - Littelmann) $V(\lambda)^a \underset{S(\mathfrak{n}_-)\text{-mods}}{\cong} \tilde{V}(\psi(\lambda))_w$

where
- $w = s_n s_{n-1} s_{n+1} s_{n-2} s_n s_{n+2} \dots s_2 s_4 s_6 \dots s_{2n-2}$
- $\psi : P^+ \longrightarrow \tilde{P}^+$, $\omega_i \longmapsto \tilde{\omega}_{2n-i}$

Let $\mu \in \tilde{P}^+ \rightsquigarrow \tilde{V}(\mu)$ irrep of hw μ

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<u>Remarks</u>

Let $\mu \in \tilde{P}^+$ ($\mathfrak{sl}_{2n}$) $\rightsquigarrow \tilde{V}(\mu)$ irrep of hw μ

$$\tilde{V}(\mu) = \bigoplus \tilde{V}(\mu)_\nu \qquad \tilde{V}(\mu)_{x\mu} \cong \mathbb{C}\cdot v_{x\mu} \quad x \in \tilde{W} = \mathfrak{S}_{2n}$$

The Demazure module $\tilde{V}(\mu)_x$ is the $\mathcal{U}(\tilde{\mathfrak{b}})$-submodule of $\tilde{V}(\mu)$ generated by $v_{x\mu}$

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• $\psi : P^+ \longrightarrow \tilde{P}^+$

$\omega_i \longmapsto \tilde{\omega}_{2n-i}$

<u>Remarks</u> ① True over $\mathbb{Z}$

Let $\mu \in \widetilde{P}^+$ $\leadsto$ $\widetilde{V}(\mu)$ irrep of hw μ

$\mathfrak{sl}_{2n}$

$$\widetilde{V}(\mu) = \bigoplus \widetilde{V}(\mu)_\nu \qquad \widetilde{V}(\mu)_{x\mu} \cong \mathbb{C}\cdot v_{x\mu} \quad x \in \widetilde{W} = \widetilde{\mathfrak{S}_{2n}}$$

The Demazure module $\widetilde{V}(\mu)_x$ is the $\mathcal{U}(\tilde{\mathfrak{b}})$-submodule of $\widetilde{V}(\mu)$ generated by $v_{x\mu}$

$$\widetilde{V}(\mu)_x \supseteq \mathcal{U}(\tilde{\mathfrak{b}}).v_{x\mu}$$

<u>Thm</u> (Cerulli Irelli - L - Littelmann) $V(\lambda)^a \underset{S(\mathfrak{n}_-)\text{-mods}}{\cong} \widetilde{V}(\psi(\lambda))_w$

where • $w = s_n s_{n-1} s_{n+1} s_{n-2} s_n s_{n+2} \dots s_2 s_4 s_6 \dots s_{2n-2}$

• $\psi : P^+ \longrightarrow \widetilde{P}^+$

$\omega_i \longmapsto \tilde{\omega}_{2n-i}$

<u>Remarks</u> ① True over $\mathbb{Z}$ ② True also in type C

Thanks!