Cluster algebras and quantum loop algebras

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Theorem (Hernandez-L 08, Nakajima, Kimura-Qin, Qin, HL 13)

Grothendieck rings of monoidal subcategories of $\mathscr{C}$ have natural cluster algebra structure

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Definition

$\mathscr{A}$, cluster algebra with initial seed (z, Γ)

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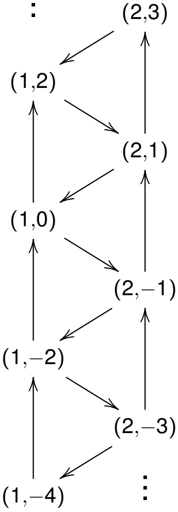

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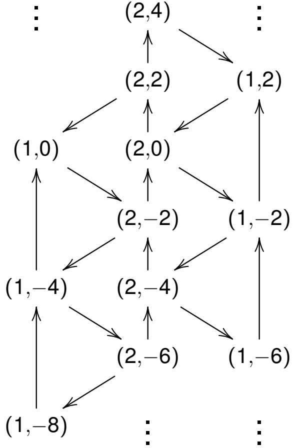

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Main observation

Under the change of variables

$$Y_{i,q^s} = \frac{z_{i,s-d_i}}{z_{i,s+d_i}}$$

the q-characters of certain simple objects of $\mathscr{C}$ become equal to certain cluster variables of $\mathscr{A}$.

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- Fundamental module $L(Y_{1,q})$:

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- Cluster variable obtained by mutation at $(1,2)$ followed by mutation at $(2,3)$:

$$(\mu_{(2,3)} \circ \mu_{(1,2)})(z_{2,3}) = \frac{z_{1,0}}{z_{1,2}} + \frac{z_{1,4} z_{2,1}}{z_{1,2} z_{2,3}} + \frac{z_{2,5}}{z_{2,3}}$$

Theorem (Hernandez-L 2013)

The "main observation" holds for all Kirillov-Reshetikhin modules:

$$W^{(i)}_{k,a} := L(Y_{i,a} Y_{i,aq^{2d_i}} \cdots Y_{i,aq^{(2k-2)d_i}})$$

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Conjecture

The q-characters of real simple objects of $\mathscr{C}$ are equal to certain cluster monomials of $\mathscr{A}$ (under the above change of variables).

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The building blocks of $\mathscr{O}$ are the prefundamental modules:

$$L^+_{i,a}, \quad L^-_{i,a}, \qquad (i \in I,\ a \in \mathbb{C}^*)$$

Prefundamental modules in type A_1

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- $U_q(\mathfrak{sl}_2)$: generators e, f, k (k invertible), relations:

$$ke = q^2 ek, \quad kf = q^{-2} fk, \quad [e, f] = \frac{k - k^{-1}}{q - q^{-1}}$$

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- $U_q(L\mathfrak{sl}_2)$: quotient of $U_q(\widehat{\mathfrak{sl}}_2)$ by $k_0 k_1 = 1$.
- for $a \in \mathbb{C}^*$, evaluation homomorphism $\mathrm{ev}_a : U_q(L\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$

$$e_1 \mapsto e, \quad f_1 \mapsto f, \quad k_1 \mapsto k, \quad e_0 \mapsto q^{-1} a^{-1} f, \quad f_0 \mapsto qae$$

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- V_n: $U_q(\mathfrak{sl}_2)$-module with basis $(v_0, v_1, \ldots, v_n)$

$$e v_i = v_{i-1}, \quad f v_i = [i+1][n-i] v_{i+1}, \quad k v_i = q^{n-2i} v_i$$

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  L^- with basis $(v_i \mid i \in \mathbb{Z}_{\geq 0})$

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  L_a^+, highest ℓ-weight $1 - au$
  L_a^-, highest ℓ-weight $\dfrac{1}{1 - au}$
- If $\lambda = b\varpi$ is an $\mathfrak{sl}_2$-weight, also have the one-dimensional $U_q(\mathfrak{b})$-module $[\lambda]$:

$$e_0 = e_1 = 0, \quad k_1 = q^b.$$

  $[\lambda]$ is a zero prefundamental module.

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- $\rightsquigarrow$ positive and negative prefundamental $U_q(\mathfrak{b})$-modules:

  L_a^+, highest ℓ-weight $1 - au$
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- If $\lambda = b\varpi$ is an $\mathfrak{sl}_2$-weight, also have the one-dimensional $U_q(\mathfrak{b})$-module $[\lambda]$:

$$e_0 = e_1 = 0, \quad k_1 = q^b.$$

  $[\lambda]$ is a zero prefundamental module.
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- $\rightsquigarrow \mathscr{C} \subset \mathscr{O}$.

Definition

- $\Psi > 0$ if it is a product of $\Psi_{i,a}^{+}$, $Y_{i,a}$, and $[\lambda]$.
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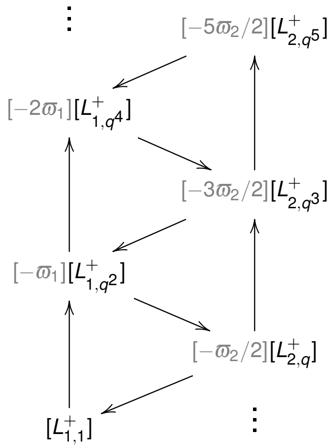
Theorem (Hernandez-L 2016)

The assignment

$$[(r/2d_i)\varpi_i] \otimes z_{i,r} \mapsto [L_{i,q^r}^{+}], \qquad ((i, r) \in V)$$

extends to an isomorphism from (a completion of) A to $K_0(\mathscr{O}_{\mathbb{Z}}^{+})$.

Example: type A_2



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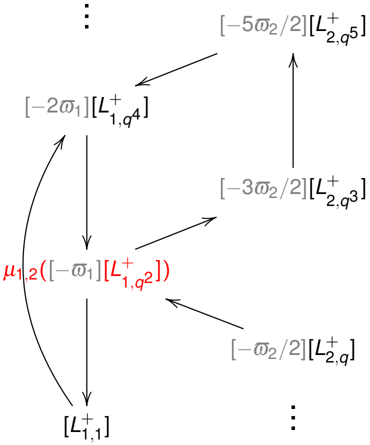

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$$= [L(Y_{1,q})]$$

Last equality follows from:

Theorem (Frenkel-Hernandez 2015)

Let $M \in \mathscr{C}$. If one replaces in the q-character of M every $Y_{i,a}$ by

$$[\varpi_i]\frac{[L^+_{i,aq^{-d_i}}]}{[L^+_{i,aq^{d_i}}]}$$

and $\chi_q(M)$ by $[M]$, one obtains a valid relation in the fraction field of $K_0(\mathscr{O})$.

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- The conjecture is true for one-step mutations for all types:

$$\mu_{i,r}(z_{i,r}) = [\cdots] \left[L\left(Y_{i,q^{r-d_i}} \prod_{j, c_{ji}<0} \Psi^+_{j,q^{r-d_j c_{j,i}}} \right) \right]$$

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Theorem (Hernandez-Frenkel 2016)

The relations given by one-step mutations yield the proof of the Bethe Ansatz equations for quantum integrable models associated with $U_q(L\mathfrak{g})$.