A recursion formula for some character values of classical groups and some applications

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Character table of S_n inside that of $\mathrm{GL}_n(q)$

Consider $G = \mathrm{GL}_n(q) \leq \mathbf{G}^F$ with $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)$.

$$\begin{aligned} \{F\text{-stable maximal tori}\}/\sim_G \quad &\stackrel{\sim}{\longleftrightarrow} \quad \{(F-)\text{conjugacy classes of } S_n\} \\ &\stackrel{\sim}{\longleftrightarrow} \quad \text{partitions } \{\lambda \vdash n\} \end{aligned}$$

$$\mathrm{Irr}(S_n) = \{\chi_\lambda \mid \ \lambda \vdash n\}$$

$s \in G$ regular semisimple of type λ $:\Leftrightarrow$ $C^0_{\mathbf{G}}(s)$ is maximal torus of type λ

$$\mathrm{Irr}(G) = \dot{\bigcup_s} \mathcal{E}(G, s)$$

Unipotent characters: $\mathcal{E}(G, 1) = \{\chi_{1,\lambda} \mid \ \lambda \vdash n\}$

Observation.

Let $t_\mu \in G$ *be regular semisimple of type* $\mu \vdash n$*,* $\pi_\mu \in S_n$ *of cycle type* μ*, and let* $\lambda \vdash n$*. Then*

$$\chi_{1,\lambda}(t_\mu) = \chi_\lambda(\pi_\mu).$$

(This holds for all q and all regular semisimple classes of that type.)

Classical groups

From now

$$G = \begin{cases} \mathrm{Spin}_{2n+1}(q) & \subset & \mathrm{Spin}_{2n+1}(\bar{\mathbb{F}}_q), & n \geq 2 & \text{(type } B_n) \\ \mathrm{Sp}_{2n}(q) & \subset & \mathrm{Sp}_{2n}(\bar{\mathbb{F}}_q), & n \geq 3 & \text{(type } C_n) \\ \mathrm{Spin}^{\pm}_{2n}(q) & \subset & \mathrm{Spin}_{2n}(\bar{\mathbb{F}}_q), & n \geq 4 & \text{(type } D_n, {}^2D_n) \end{cases}$$

$$\{F\text{-conjugacy classes of Weyl group } W\} \quad \overset{\sim}{\longleftrightarrow} \quad \{(\lambda, \mu) \vdash n\}$$

(only μ with even/odd number of entries in types D_n, ${}^2D_n$; degenerate cases in D_n)

$$\mathcal{E}(G, 1) \quad \overset{\sim}{\longleftrightarrow} \quad \{\chi_{\mathcal{S}} \mid \mathcal{S} \text{ certain symbols of rank } n\}$$

Symbols

$$S = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \lambda_1 < \lambda_2 < \ldots < \lambda_r \\ \mu_1 < \mu_2 < \ldots < \mu_s \end{pmatrix}$$

$$\text{"="} \begin{pmatrix} Y \\ X \end{pmatrix} \text{"="} \begin{pmatrix} 0 < \lambda_1 + 1 < \lambda_2 + 1 < \ldots < \lambda_r + 1 \\ 0 < \mu_1 + 1 < \mu_2 + 1 < \ldots < \mu_s + 1 \end{pmatrix}$$

with defect $|r - s|$ odd (B_n, C_n), $0 \mod 4$ (D_n), $2 \mod 4$ (${}^2D_n$)

(degenerate symbols $\begin{pmatrix} X \\ X \end{pmatrix}$ correspond to two characters, χ_S denotes their sum)

Removing d-hook h from $\mathcal{S}$, $\mathcal{S} \setminus h$: Remove an entry λ_i from X such that $\lambda_i \geq d$ and $\lambda_i - d$ is not in X and add $\lambda_i - d$ to X. And the same for Y.

Removing d-cohook c from $\mathcal{S}$, $\mathcal{S} \setminus c$: Remove an entry λ_i from X such that $\lambda_i \geq d$ and $\lambda_i - d$ is not in Y and add $\lambda_i - d$ to Y. And the same for X, Y interchanged.

If $\mathcal{S}$ is of rank n, h a d-hook, c a d-cohook then $\mathcal{S} \setminus h$ and $\mathcal{S} \setminus c$ have rank $n - d$.

Murnaghan-Nakayama formula for classical groups

In this theorem G can be any group of type B_n, C_n, D_n, ${}^2D_n$.

Theorem

Let $t \in G$ be regular semisimple of type (λ, μ).

(a) *If λ contains d then*

$$\chi_{\mathcal{S}}(t) = \sum_h \varepsilon_h \chi_{\mathcal{S} \setminus h}(t),$$

where the sum is over all d-hooks h of $\mathcal{S}$.

(b) *Similar statement with μ contains d and cohooks . . .*

ε_h is a sign associated to h.

$\mathcal{S} \setminus h$ is symbol of unipotent character of Levi subgroup of rank $n - d$, (centralizer of torus of order $q^d - 1$).

$t \in L$ is regular semisimple of type $(\lambda \setminus d, \mu)$.

Proof. Character formula for ${}^*R_L^G$; theorem of Asai on decomposition of ${}^*R_L^G(\chi_{\mathcal{S}})$ where L is centralizer of $(q^d \pm 1)$-torus.

Regular semisimple elements

Lemma

$G \in \{Sp_{2n}(q), Spin_{2n+1}(q), Spin^{\pm}_{2n}(q)\}$ contains regular semisimple elements of type $((\lambda_1, \ldots, \lambda_r), (\mu_1, \ldots, \mu_s))$ in these cases:

(a) *$q > 3$, $\lambda_1 < \lambda_2 < \cdots < \lambda_r$, $\mu_1 < \mu_2 < \cdots < \mu_s$.*

(b) *$q \in \{2, 3\}$, $\lambda_1 < \lambda_2 < \cdots < \lambda_r$, $\mu_1 < \mu_2 < \cdots < \mu_s$ with $\lambda_i \neq 2$ (type D_n, ${}^2D_n$)), resp. $\lambda_i \notin \{1, 2\}$ (type B_n, C_n).*

(c) *In type D_n: $2 < \lambda_1 < \cdots < \lambda_r$ and $1 = \mu_1 = \mu_2 < \mu_3 < \cdots < \mu_s$.*

ℓ-singular elements

ℓ: odd prime, $\ell \nmid q$

$d =$ order of $q \mod \ell$ (so $\ell \mid (q^d - 1)$)

$$\begin{array}{ll} \text{If } d \text{ odd:} & \ell \mid (q^k - 1) \Leftrightarrow d \mid k \\ \text{If } d = 2e \text{ even:} & \ell \mid (q^k - 1) \Leftrightarrow d \mid k \\ & \ell \mid (q^k + 1) \Leftrightarrow e \mid k \text{ and } k/e \text{ odd} \end{array}$$

$T \leq G$ maximal torus of type $((\lambda_1, \ldots, \lambda_r), (\mu_1, \ldots, \mu_s))$, then

$$|T| = \prod_{\lambda_i}(q^{\lambda_i} - 1) \prod_{\mu_i}(q^{\mu_i} + 1)$$

Zeroes of characters

Now assume: the Sylow-ℓ-subgroup of G is not cyclic

Theorem

Let $\chi \in Irr(G)$ such that $\chi(t) \neq 0$ on all ℓ-singular regular semisimple elements t. Then one of the following holds:

(a) *If χ is not unipotent then $G \in \{Sp_4(2), Sp_8(2)\}$.*

(b) *χ is the trivial or Steinberg character.*

(c) *χ or its Alvis-Curtis dual is a unipotent character $\chi_{\mathcal{S}}$:*

- (1) *$G = Sp_{2n}(q)$ or $G = Spin_{2n+1}(q)$, d odd, $n = 2d + r$, $0 \leq r < d$ and*

$$\mathcal{S} = \begin{pmatrix} & 1 & 2 & \dots & d-r-1 & d & 2d \\ 0 & 1 & & \dots & d-r-1 & & \end{pmatrix}.$$

- (2)-(6) *Similar for other types, d odd or even.*
- (7)-(10) *Cases for $q = 2$, e.g., $G = Sp_6(2)$, $d = 2$ and $\mathcal{S} = \begin{pmatrix} 013 \\ - \end{pmatrix}$.*

Classifying ℓ-modular endotrivial modules

A simple ℓ-modular endotrivial representation of G can be lifted to a characteristic 0 representation with character χ such that

- $|\chi(g)| = 1$ for all ℓ-singular elements $g \in G$
- $\chi(1) \equiv \pm 1 \mod |G|_\ell$
- χ is irreducible modulo ℓ

Theorem

Let G and ℓ be as before. If $\chi \in Irr(G)$ is the lifted character of a non-trivial simple ℓ-modular endotrivial module then $G = Sp_8(2)$, $\chi = \chi_S$ with $\mathcal{S} = \begin{pmatrix} 01 \\ 4 \end{pmatrix}$.

Constituents of 1-PIMs

(Answering a question of Külshammer, Koshitani, Sambale.)

Theorem

Let G, ℓ be as before. Then the ℓ-modular projective cover of the trivial character has at least three constituents.

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