Towards an Exotic Robinson-Schensted Correspondence

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Nilpotent Orbits in Type A

We always work over $\mathbb{C}$. The Nilpotent Cone for GL_n is:

$$\mathcal{N}(\mathfrak{gl}_n) := \{x \in \mathrm{Mat}_n \mid x \text{ is nilpotent, i.e. all eigenvalues are } 0\},$$

where GL_n acts by conjugation: $g \cdot x := gxg^{-1}$

Theorem (Jordan Canonical Form)

GL_n-orbits on $\mathcal{N}(\mathfrak{gl}_n)$ are classified by partitions of n. For $\lambda \in \mathcal{P}_n, \mathcal{O}_\lambda$ consists of those $x \in \mathcal{N}(\mathfrak{gl}_n)$ whose Jordan blocks have size $(\lambda_1, \lambda_2, \ldots) = \lambda$. Say x has Jordan Type λ

The G-orbits stratify $\mathcal{N}$ with the following closure ordering:

Closure Orderings in Type A

We have $\mathcal{N} = \sqcup_{\lambda \in \mathcal{P}_n} \mathcal{O}_\lambda$ and

$$\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda} \iff \mu \trianglelefteq \lambda \iff \begin{array}{rl} \mu_1 & \leq \lambda_1 \\ \mu_1 + \mu_2 & \leq \lambda_1 + \lambda_2 \\ \vdots & \vdots \quad \vdots \end{array}$$

The symmetric group on n-letters, S_n is the Weyl group of GL_n.
(In general $W = N_G(T)/T$, where T is a maximal torus).
We have the following bijections:

$$\begin{array}{ccccc} GL_n \backslash \mathcal{N}(\mathfrak{gl}_n) & \stackrel{\sim}{\longleftrightarrow} & \mathcal{P}_n & \stackrel{\sim}{\longleftrightarrow} & \text{Irreps of } S_n \\ GL_n \cdot \mathrm{x} = \mathcal{O}_\lambda & \longleftrightarrow & \lambda & \longleftrightarrow & S^\lambda \\ & & & & \text{(Specht Module)} \end{array}$$

'Combinatorial' Springer correspondence in Type A

In general, we have an injection:

$$G \backslash \mathcal{N} \hookrightarrow \mathrm{Irr}(W).$$

Nilpotent Orbits in Types B and C

The groups and respective nilpotent cones are:

$$\begin{aligned} SO_{2n+1} &= \{g \in \mathrm{Mat}_{2n+1} \mid \det(g) = 1, \ (gv, gw) = (v, w), \ \forall v, w\}, \\ \mathcal{N}(\mathfrak{so}_{2n+1}) &= \{\mathrm{x} \in \mathrm{Mat}_{2n+1} \mid \mathrm{x} \text{ nilpotent}, \ (\mathrm{x}v, v) = 0, \ \forall v\}, \end{aligned}$$

where $(,)$ is a symmetric non-degenerate bilinear form on $\mathbb{C}^{2n+1}$.

$$\begin{aligned} Sp_{2n} &= \{g \in \mathrm{Mat}_{2n} \mid \det(g) = 1, \ \langle gv, gw \rangle = \langle v, w \rangle, \ \forall v, w\}, \\ \mathcal{N}(\mathfrak{sp}_{2n}) &= \{\mathrm{x} \in \mathrm{Mat}_{2n} \mid \mathrm{x} \text{ nilpotent}, \ \langle \mathrm{x}v, w \rangle = \langle \mathrm{x}w, v \rangle, \ \forall v, w\}, \end{aligned}$$

where $\langle , \rangle$ is a symplectic non-degenerate bilinear form on $\mathbb{C}^{2n}$.

Theorem (Wall, Gerstenhaber, Hesselink)

Nilpotent orbits of SO_{2n+1} and Sp_{2n} are classified by Jordan Type:

$$SO_{2n+1} \backslash \mathcal{N}(\mathfrak{so}_{2n+1}) \stackrel{\sim}{\longleftrightarrow} \left\{ \lambda \vdash 2n+1 \,\middle|\, \begin{array}{c} \textit{even parts occur with} \\ \textit{even multiplicity} \end{array} \right\},$$

$$Sp_{2n} \backslash \mathcal{N}(\mathfrak{sp}_{2n}) \stackrel{\sim}{\longleftrightarrow} \left\{ \lambda \vdash 2n \,\middle|\, \begin{array}{c} \textit{odd parts occur with} \\ \textit{even multiplicity} \end{array} \right\}.$$

Types B and C

SO_{2n+1} and Sp_{2n} share the same Weyl group:

$$W(B_n) = W(C_n) := \{\pm 1\} \wr S_n.$$

Irreducible representations of $W(C_n)$ are classified by bipartitions of n:

$$\mathcal{Q}_n = \{\lambda := (\mu;\nu) : \mu, \nu \text{ partitions}; \, |\mu| + |\nu| = n\}.$$

Theorem (Lusztig, Shoji)

The nilpotent orbits in Types B and C correspond to the following irreducible representations:

$$\begin{aligned} SO_{2n+1} \backslash \mathcal{N}(\mathfrak{so}_{2n+1}) \quad &\longleftrightarrow \quad \{(\mu;\nu) \,|\, \mu_i \geq \nu_i - 2, \nu_i \geq \mu_{i+1}\}, \\ Sp_{2n} \backslash \mathcal{N}(\mathfrak{sp}_{2n}) \quad &\longleftrightarrow \quad \{(\mu;\nu) \,|\, \mu_i \geq \nu_i - 1, \nu_i \geq \mu_{i+1}\}. \end{aligned}$$

Types B and C

Closure Orderings in Types B and C

For (ρ, σ) and $(\mu, \nu) \in \mathcal{Q}_n$, we have:

$$\mathcal{O}_{(\rho,\sigma)} \subseteq \overline{\mathcal{O}_{(\mu,\nu)}} \iff \begin{array}{cl} \rho_1 & \leq \mu_1 \\ \rho_1 + \sigma_1 & \leq \mu_1 + \nu_1 \\ \rho_1 + \sigma_1 + \rho_2 & \leq \mu_1 + \nu_1 + \mu_2 \\ \rho_1 + \sigma_1 + \rho_2 + \sigma_2 & \leq \mu_1 + \nu_1 + \mu_2 + \nu_2 \\ \vdots & \vdots \quad \vdots \end{array}$$

An Explicit W-action

There is a W-action on the Springer Sheaf:

$$\underline{\mathsf{Spr}} = \pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] \in \mathrm{Perv}_G(\mathcal{N}, \mathbb{C}) \subset D^b_c(\mathcal{N})$$

where $\pi : \widetilde{\mathcal{N}} \longrightarrow \mathcal{N}$ is the Springer resolution. By the decomposition theorem:

$$\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] = \bigoplus_{\substack{\mathcal{O} \in \mathcal{N}/G \\ \mathcal{E} \in \mathrm{Loc}_{\mathbb{C}}(\mathcal{O})}} IC(\mathcal{O}, \mathcal{E}) \otimes \mathrm{Hom}_{\mathrm{Perv}_G(\mathcal{N},\mathbb{C})}(\underline{\mathsf{Spr}}, IC(\mathcal{O}, \mathcal{E}))$$

Springer Miracles!

1. $\mathrm{Hom}_{\mathrm{Perv}_G(\mathcal{N},\mathbb{C})}(\underline{\mathsf{Spr}}, IC(\overline{\mathcal{O}}, \mathcal{E})) \cong H^{top}(\pi^{-1}(x))$, for $x \in \mathcal{O}$.
2. $H^{top}(\pi^{-1}(x))$ carries a W-action.

Taking endomorphisms:

$$\mathrm{End}(\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]) = \bigoplus_{\mathcal{O} \in \mathcal{N}/G,\ \mathcal{E} \in \mathrm{Loc}_{\mathbb{C}}(\mathcal{O})} \mathrm{End}(H^{top}(\pi^{-1}(x))) \cong \mathbb{C}W.$$

A Word on Local Systems

- The local systems $\mathcal{E} \in \mathrm{Loc}(\mathcal{O})$ correspond to representations of the component group $A(\mathcal{O}) = C_G(x)/C_G(x)^0$, for $x \in \mathcal{O}$.
- In Type A, GL_n all centralisers are connected, so no non-trivial local systems appear in the decomposition.
- In Type C, some non-trivial local systems occur.
- In 2009, Kato define an Exotic Nilpotent Cone for which Sp_{2n} acts with connected stablisers.

Kato's Exotic Nilpotent Cone

Let $\mathcal{S}$ be the Sp_{2n}-invariant complement of $\mathfrak{sp}_{2n}$ in $\mathfrak{gl}_{2n}$; i.e.

$$\mathcal{S} = \{x \in \mathrm{Mat}_{2n}(\mathbb{C}) \mid \langle xv, w\rangle = \langle v, xw\rangle, \forall v, w \in \mathbb{C}^{2n}\}.$$

Define $\mathcal{N}(\mathcal{S}) := \mathcal{S} \cap \mathcal{N}(\mathfrak{gl}_{2n})$.

Definition (Kato, 2009)

The Exotic Nilpotent Cone of Type C is

$$\mathfrak{N} = \mathbb{C}^{2n} \times \mathcal{N}(\mathcal{S}) = \{(v, x) \mid v \in \mathbb{C}^{2n}, x \in \mathcal{N}(\mathcal{S})\}.$$

The exotic nilpotent cone recovers the 'Springer Miracles':

Theorem (Kato, 2009)

- $\mathrm{End}(\psi_* \underline{\mathbb{C}}_{\widetilde{\mathfrak{N}}}) \cong \mathbb{C}W(C_n)$;
- $Sp_{2n} \backslash \mathfrak{N} \longleftrightarrow \mathrm{Irr}(W(C_n)) \longleftrightarrow \mathcal{Q}_n$.

Sp_{2n}-orbits on the exotic nilpotent cone

Let $\mathbb{O}_{(\mu,\nu)}$ be the Sp_{2n}-orbit corresponding to $(\mu, \nu) \in \mathcal{Q}_n$.

Theorem (Achar-Henderson, 2009)

For (ρ, σ) and $(\mu, \nu) \in \mathcal{Q}_n$, we have:

$$\mathbb{O}_{(\rho,\sigma)} \subseteq \overline{\mathbb{O}_{(\mu,\nu)}} \iff \begin{array}{cl} \rho_1 & \leq \mu_1 \\ \rho_1 + \sigma_1 & \leq \mu_1 + \nu_1 \\ \rho_1 + \sigma_1 + \rho_2 & \leq \mu_1 + \nu_1 + \mu_2 \\ \rho_1 + \sigma_1 + \rho_2 + \sigma_2 & \leq \mu_1 + \nu_1 + \mu_2 + \nu_2 \\ \vdots & \vdots \qquad \vdots \end{array}$$

This was prove using their work on the enhanced nilpotent cone for GL_{2n}; $\mathbb{C}^{2n} \times \mathcal{N}(\mathfrak{gl}_{2n})$

Why does the Exotic Nilpotent Cone Exist?

- An accident of the root system of type C;
- In characteristic 2, the adjoint representation of $Sp_{2n}(\overline{\mathbb{F}}_2)$ is reducible with two constituents;
- the weights of these constituents are "short roots" for one and the "long roots" for the other;
- the short root constituent is $\Lambda^2(\mathbb{F}_2^{2n})$;
- the "long root" constituent is a Frobenius twist of the natural module whose $\mathbb{F}_2^{2n}$ weights are half the long roots.

In characteristic 0, the exotic nilpotent cone is the Hilbert nullcone of the representation $\mathbb{C}^{2n} \oplus \Lambda^2(\mathbb{C}^{2n}) := \mathbb{V}$.

Springer Fibres

Let $B \subseteq G$ be a Borel subgroup. Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{b} := \mathrm{Lie}(B)$. In general the Springer resolution has the form:

$$\pi : G \times_B \mathfrak{n} \longrightarrow \mathcal{N}; (g, \mathrm{x}) \mapsto g\mathrm{x}g^{-1}$$

where $G \times_B \mathfrak{n} = (G \times \mathfrak{n})/\{(g, \mathrm{x}) \sim (gb^{-1}, b\mathrm{x}b^{-1}), \forall b \in B\}$. We have $G \times_B$ is smooth and π is proper, and general fibres $\mathcal{F}_\mathrm{x} := \pi^{-1}(x)$ have the form:

(in general)	$\{\text{Borel subalgebras: } \mathfrak{b}' \subset \mathfrak{g} \mid \mathrm{x} \in \mathfrak{b}'\} \subseteq G/B := \mathcal{B}$
(Type A)	$\{0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n \mid \dim(V_i) = i, \mathrm{x}(V_i) \subset V_{i-1}\}$
(Type C)	$\{0 \subset V_1 \subset \ldots \subset V_{2n} = \mathbb{C}^{2n} \mid V_{2n-i}^{\perp} = V_i, \, \mathrm{x}(V_i) \subset V_{i-1}\}$

In the exotic case, the resolution looks like:

$$\psi : G \times_B \mathbb{V}^{\geq 0} \longrightarrow \mathfrak{N}$$

with fibres above a point (v, x):

$$\mathcal{C}_{(v,\mathrm{x})} := \{0 \subset V_1 \subset \ldots \subset \mathbb{C}^{2n} \mid V_{2n-i}^{\perp} = V_i, \; v \in V_n, \; \mathrm{x}(V_i) \subset V_{i-1}\}.$$

Let $V = \mathbb{C}^n$, fix $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}_n$ and $\mathsf{x} \in \mathcal{N}$ of type λ. Let $\mathcal{F}_\mathsf{x}$ be the fibre of x. Recall that for S^λ a Specht module for S_n, the set

$$\mathsf{Std}(\lambda) := \{\text{standard tableau of shape } \lambda\},$$

labels a basis for S^λ.

Theorem (Spalenstein, 1976)

There is a map (defined inductively)

$$\Theta : \mathcal{F}_\mathsf{x} \longrightarrow \mathsf{Std}(\lambda),$$

which induces a bijection:

$$\mathsf{Irr}(\mathcal{F}_\mathsf{x}) \stackrel{\sim}{\longleftrightarrow} \mathsf{Std}(\lambda).$$

Springer Fibres and Standard Young Tableaux

For $T \in \mathrm{Std}(\lambda)$, let $\mathcal{F}_T := \Theta^{-1}(T)$
(induction shows that the $\mathcal{F}_T$ are non-empty for all $T \in \mathrm{Std}(\lambda)$).
Therefore

$$\mathcal{F}_x = \bigsqcup_{T \in \mathrm{Std}(\lambda)} \mathcal{F}_T.$$

Theorem (Spaltenstein, 1976)

The $\mathcal{F}_T$ are:

(a) *locally closed in $\mathcal{F}_x$;*

(b) *are irreducible and all of the same dimension*

$$\dim(\mathcal{F}_T) = \sum_{i \geq 1} \frac{\lambda_i^{tr}(\lambda_i^{tr} - 1)}{2}.$$

The Steinberg Variety: Type A

Let $G = GL_n$, $\mathcal{N} = \mathcal{N}(\mathfrak{gl}_n)$ and $\mathcal{B} = G/B$ for B a Borel. Define the Steinberg variety as

$$\mathcal{Z} = \{(\mathrm{x}, F_\bullet, G_\bullet) \,|\, \mathrm{x} \in \mathcal{N}, F_\bullet, G_\bullet \in \mathcal{F}_\mathrm{x}\} \subseteq \mathcal{N} \times \mathcal{B} \times \mathcal{B}.$$

It comes with two natural projections:

$$\mathcal{N} \xleftarrow{p_1} \mathcal{Z} \xrightarrow{p_2} \mathcal{B} \times \mathcal{B},$$

which gives two ways to parametrise its irreducible components:

- by irreducible components of $\mathcal{F}_\mathrm{x}$ (i.e. by $\mathrm{Std}(\lambda)) \times \mathrm{Std}(\lambda)$): Suppose $S, T \in \mathrm{Std}(\lambda)$. Then a generic point $(\mathrm{x}, F_\bullet, G_\bullet) \in \mathcal{Z}_{(S,T)} \subseteq \mathcal{Z}$ has the property that $\Theta(F_\bullet) = S$ and $\Theta(G_\bullet) = T$.
- by elements of S_n (or GL_n-orbits on $\mathcal{B} \times \mathcal{B}$): Let $\sigma \in S_n$. Then a generic point $(\mathrm{x}', F'_\bullet, G'_\bullet) \in \mathcal{Z}_\sigma$ has the property that $F'_\bullet = \sigma(G'_\bullet)$.

The Steinberg Variety: Type A

These two ways of parametrising irreducible components of $\mathcal{Z}$ give rise to a bijection

$$S_n \stackrel{\sim}{\longleftrightarrow} \mathrm{Std}(\lambda) \times \mathrm{Std}(\lambda)$$

Theorem (Steinberg, 1988)

This bijection is an occurrence of the Robinson-Schensted Correspondence for the symmetric group.

Our Question

How much of this carries through for the exotic nilpotent cone?

- What are the irreducible components of the exotic Springer fibres?
- Parametrise irreducible components of the 'exotic' Steinberg variety.

Let (v, x) be a representative for the orbit $\mathbb{O}_{(\mu,\nu)} \subseteq \mathfrak{N}$; we say that (v, x) has Exotic type $\lambda := (\mu, \nu)$; write $\mathrm{eType}(v, x) = (\mu, \nu)$.
Let $\mathcal{C}_{(v,\mathrm{x})}$ denote the fibre $\psi^{-1}(v, \mathrm{x})$. Here we want a bijection

$$\mathrm{Irr}(\mathcal{C}_{(v,\mathrm{x})}) \longleftrightarrow \mathrm{Std}(\mu, \nu).$$

There is a natural map:

$$\begin{aligned} \Phi : \mathcal{C}_{(v,\mathrm{x})} &\longrightarrow \mathcal{Q}_1 \times \mathcal{Q}_2 \times \ldots \times \mathcal{Q}_{n-1} \times \mathcal{Q}_n \\ F_\bullet &\mapsto (\ldots, \mathrm{eType}(v + V_{n-i}, \mathrm{x}_{|V_{n-i}^{\perp}/V_{n-i}})_{i=1}^{n-1}, \ldots, (\mu, \nu)), \end{aligned}$$

where

- $(v + V_i, \mathrm{x}_{|V_{n-i}^{\perp}/V_{n-i}})$ is representative of the $Sp_{2(n-i)}$-orbit in $\mathfrak{N}_{n-i}$; and
- $\mathrm{eType}(v + V_{n-i}, \mathrm{x}_{|V_{n-i}^{\perp}/V_{n-i}}) \in \mathcal{Q}_i$.

$$\mathrm{im}(\Phi) = ??$$

Exotic Springer Fibres

Consider the set of nested sequences

$$\mathbb{T}_{(\mu,\nu)} := \left\{ (\mu_1,\nu_1) \preceq \ldots \preceq (\mu_n,\nu_n) = (\mu,\nu) \,\middle|\, \substack{(\mu_i,\nu_i)\in\mathcal{Q}_i \\ \mu_{i-1}=\mu_i,\,\nu_{i-1}<\nu_i \text{ or vice versa}} \right\}$$

For

$$\Phi : \mathcal{C}_{(v,x)} \longrightarrow \mathcal{Q}_1 \times \mathcal{Q}_2 \times \ldots \times \mathcal{Q}_{n-1} \times \mathcal{Q}_n,$$

$\mathrm{im}(\Phi)$ is not just the set of nested sequences, but....

Theorem (Nandakumar-Rosso-S'15)

Let $T_{(\mu,\nu)} \in \mathbb{T}_{(\mu,\nu)}$. The preimage $\Phi^{-1}(T_{(\mu,\nu)})$ is an irreducible sub-variety of $\mathcal{C}_{(v,x)}$ of dimension

$$b(\mu,\nu) := |\nu| + 2\sum_{i\geq 1}(i-1)(\mu_i+\nu_i).$$

Exotic Springer Fibres

Not all points in the exotic Springer fibre $\mathbb{C}_{(\mu,\nu)}$ map to nested sequences of bipartitions, but we have:

Lemma (Nadakumar-Rosso-S'15)

Let $F_\bullet \in \mathcal{C}_{(v,x)}$ *such that* $\Phi(F_\bullet) \notin \mathbb{T}_{(\mu,\nu)}$. *Then* $F_\bullet \in \overline{\Phi^{-1}(T_{(\mu,\nu)})}$ *for some* $T_{(\mu,\nu)} \in \mathbb{T}_{(\mu,\nu)}$

Corollary

Taking closures of preimages of nested sequences, we have

$$\mathcal{C}_{(v,x)} = \bigcup_{T_{(\mu,\nu)} \in \mathbb{T}_{(\mu,\nu)}} \overline{\Phi^{-1}(T_{(\mu,\nu)})}.$$

Exotic Springer Fibres and Combinatorics

Theorem (Nadakumar-Rosso-S'15)

For $T_{(\mu,\nu)} \in \mathbb{T}_{(\mu,\nu)}$, the $\Phi^{-1}(T_{(\mu,\nu)})$ are locally closed, irreducible sub-varieties of $\mathcal{C}_{(\mu,\nu)}$ of equal dimension, and so

$$\mathsf{Irr}(\mathcal{C}_{(\mu,\nu)}) \overset{\sim}{\longleftrightarrow} \mathsf{Std}(\mu,\nu).$$

Work in Progress

An exotic Robinson-Schensted-Correspondence for $W(C_n)$.

- If $\lambda = (-;\nu)$, then the algorithm is a 'transpose' of the RS-algorithm in type A.
- If $\lambda = (\mu;-)$, then algorithm is easy to describe.
- If $\lambda = (\mu;\nu)$, then it gets complicated!