On Extensions of Hook Weyl modules

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Polynomial Representation of GL_n

<u>Notation</u>: λ = partition

$\Delta(\lambda)$ = Weyl module

$\nabla(\lambda)$ = Costandard (or Schur) module

$L(\lambda)$ = Simple module

Main open problem:

The determination of $\mathrm{ch} L(\lambda)$

$M_{\mathbb{K}}(n,r)$:= finite-dimensional homogeneous of degree r polynomial representations of $GL_n(\mathbb{K})$

($\mathbb{K}$ = field or $\mathbb{Z}$)

The problem is solved in some special cases:

- λ, μ have at most two parts, $i=1$

~ **Akin** and **Buchsbaum**, Characteristic-free representation theory of the general linear group II: Homological considerations, Adv. in Math. 72 (1988), 171-210. Integral

~ **Buchsbaum**, Aspects of characteristic-free representation theory of GL_n and some applications to intertwining numbers, Acta Appl. Math. 21 (1990), 247-261. Integral

~ **Erdmann**, Ext^1 for Weyl modules of $SL_2(K)$, Mathematische Zeitschrift 2 (1994), 447-459. Modular

- λ, μ are hook partitions, $\mu - \lambda$ = positive root, $i=1$

~ **Maliakas**, Resolutions, homological dimensions and extesions of hook representations, Commun. Algebra, 19 (1991), 2195-2216 Integral

- λ, μ are p-hook partitions for every i

~ **Doty** and **Martin**, Hook modules for general linear groups, Arch. Math 92 (2009), 206-214 Modular

- $\mu - \lambda$ = positive root, $i=1$

~ **Kulkarni**, On the Ext groups for Weyl modules of GL_n, J. Algebra 304 (2006), 510-542 Integral

- λ, μ have at most 3 parts, $i=1$

~ **Buchsbaum** and **Flores**, Intertwining numbers: the three rowed case, J. Algebra 183 (1996), 605-635 Integral

- λ, μ have at most 2 parts, $\forall i$

~ Parker, Higher extensions between modules for SL_2, Adv. Math. 209 (2007) no.1, 381-405 Modular

- λ, μ have at most 3 parts and $i=0$

~ Cox and Parker, Homomorphisms between Weyl modules for SL_3, Transaction AMS, 358, 4159-4207

Modular

It is known that

$$\mathrm{ch}\,L(\lambda) = \sum (-1)^i \dim \mathrm{Ext}^i(L(\lambda), \Delta(\mu)) \cdot s_\mu$$

where $s_\mu = \mathrm{ch}\,\Delta(\mu)$ is the Schur function and the Ext groups are taken over $M_{\mathbb{K}}(n,r$

$\mathbb{K}$ is a field,

and

$$\mathrm{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \mathrm{Ext}^i(L(\nu), \Delta(\mu)) \neq 0$$

where $L(\nu)$ is quotient in a composition series of $\Delta(\lambda)$.

So, we are interested in the determination of $\mathrm{Ext}^i(\Delta(\lambda), \Delta(\mu))$

Why Hooks?

Hook Weyl modules appear as duals of the cycles in the Koszul complex:

$$0 \to \cdots \to \Lambda^{a+1} V \otimes S_{b-1} V \to \Lambda^{a} V \otimes S_{b} V \to$$
$$\to \Lambda^{a-1} V \otimes S_{b+1} V \to \cdots \to 0$$

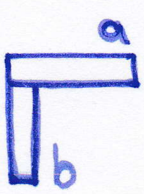
V = the natural GL_n-module

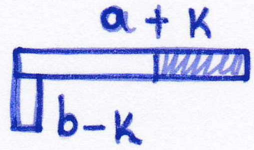
General fact:

If $\mathrm{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0$

$\Rightarrow$ $\mu > \lambda$ in dominance order

So here we examine the case where

$\lambda =$ 

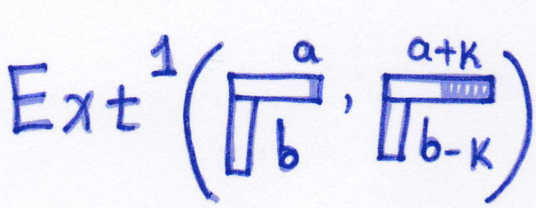
and $\mu =$ 

We will consider the corresponding integral Ext-groups $(GL_n(\mathbb{Z}))$.

We then obtain the modular Ext-groups via the

"Universal Coefficient Theorem"

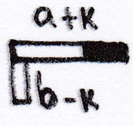
Theorem:



$$= \begin{cases} \mathbb{Z}_{a+b} & K=1 \\ \mathbb{Z}_{\frac{2}{hcf(2,a+b-K)}} & K \geq 2 \end{cases}$$

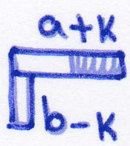
The case $K=1$ is solved (Kulkarni)

Theorem :

Ext^2( , ) =

$$\begin{cases} 0 & k=1 \\ \mathbb{Z}_{\frac{a+b}{hcf(2,a+b)}} & k=2 \\ \mathbb{Z}_{\frac{3}{hcf(3,a+b)}} & k=3 \\ \mathbb{Z}_{hcf(3,a+b)} & k=4 \\ 0 & k\geq 5 \end{cases}$$

Theorem :

Ext^k( , ) $\cong \mathbb{Z}_{d_k}$

where

$$d_k = hcf\left(\binom{a+b}{1}, \binom{a+b}{2}, \ldots, \binom{a+b}{k}\right)$$

- Our tools :

1) Projective resolutions for hook Weyl modules over the integral Schur algebra $S_{\mathbb{Z}}(n,r)$

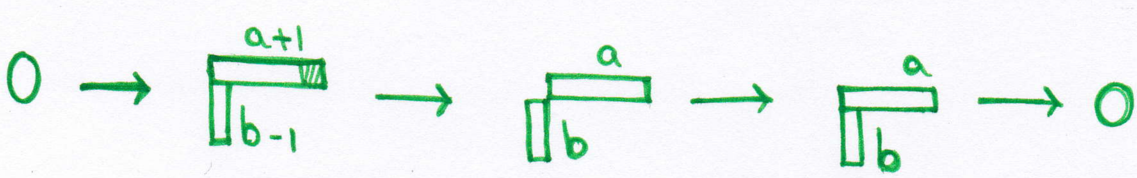
(Maliakas, Resolutions, homological dimensions and extensions of hook representations, Commun. Algebra 19, (1991), 2195-2216)

2) Long exact sequence obtained from

$$0 \to \Delta(a+1, 1^{b-1}) \to \Lambda^b V \otimes D_a V \to \Delta(a, 1^b) \to 0$$

(where DV = the devided power algebra of V)

and the functor $\mathrm{Hom}_{S_{\mathbb{Z}}}(\Delta(\lambda), -)$



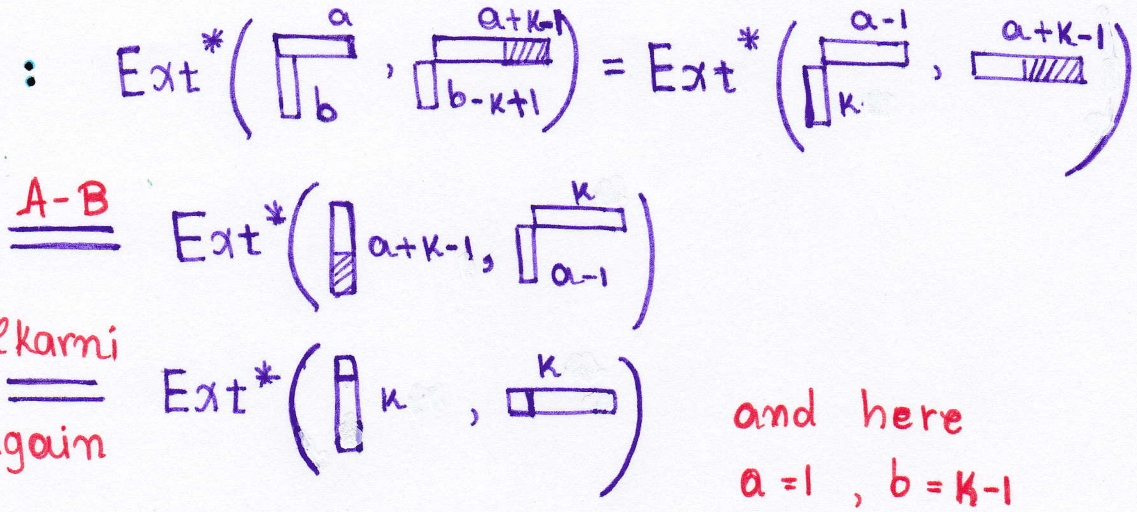
3) $\mathrm{Ext}^i_{S_{\mathbb{Z}}}(\Delta(\lambda), \Lambda^a V \otimes D_b V) = \mathrm{Ext}^i_{S_{\mathbb{Z}}}(\Lambda^q V, D_q V)$

for appropriate $q \in \mathbb{N}^*$

(Kulkarni, Skew Weyl modules for GL_n and degree reduction for Schur algebras, J. Algebra 224 (2000), 248-262.)

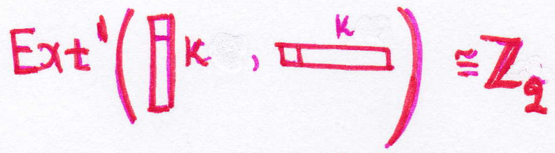
4) Computations with weight spaces of Weyl modules

Kulkarni :
corollary

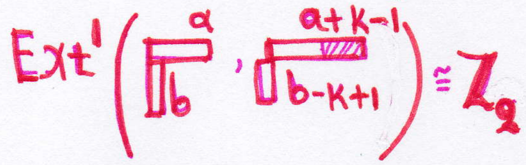


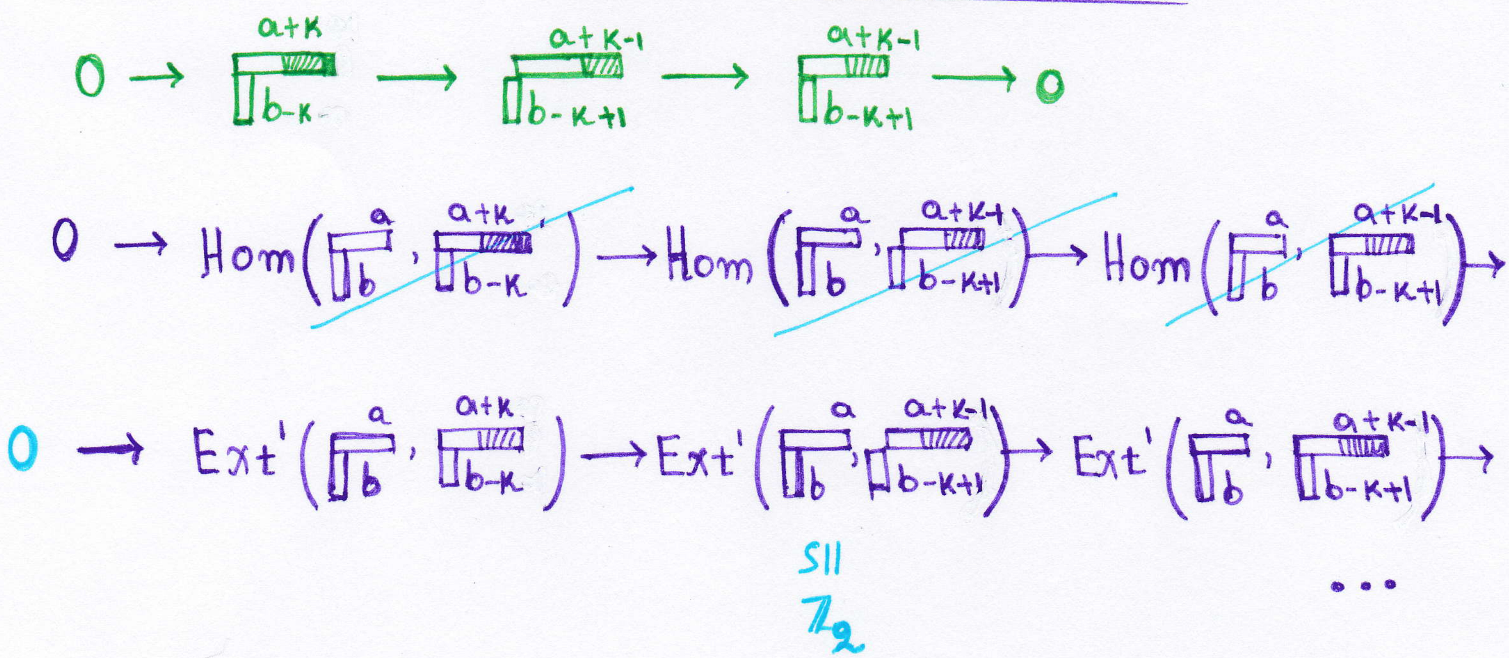
and here
$a = 1$, $b = k-1$

When $* = 1$ $\Rightarrow$

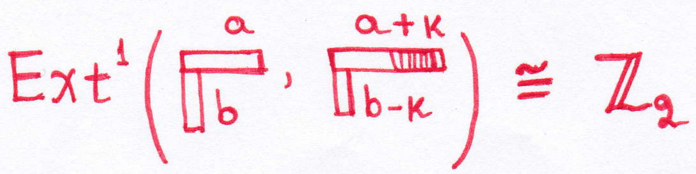


or equivalently



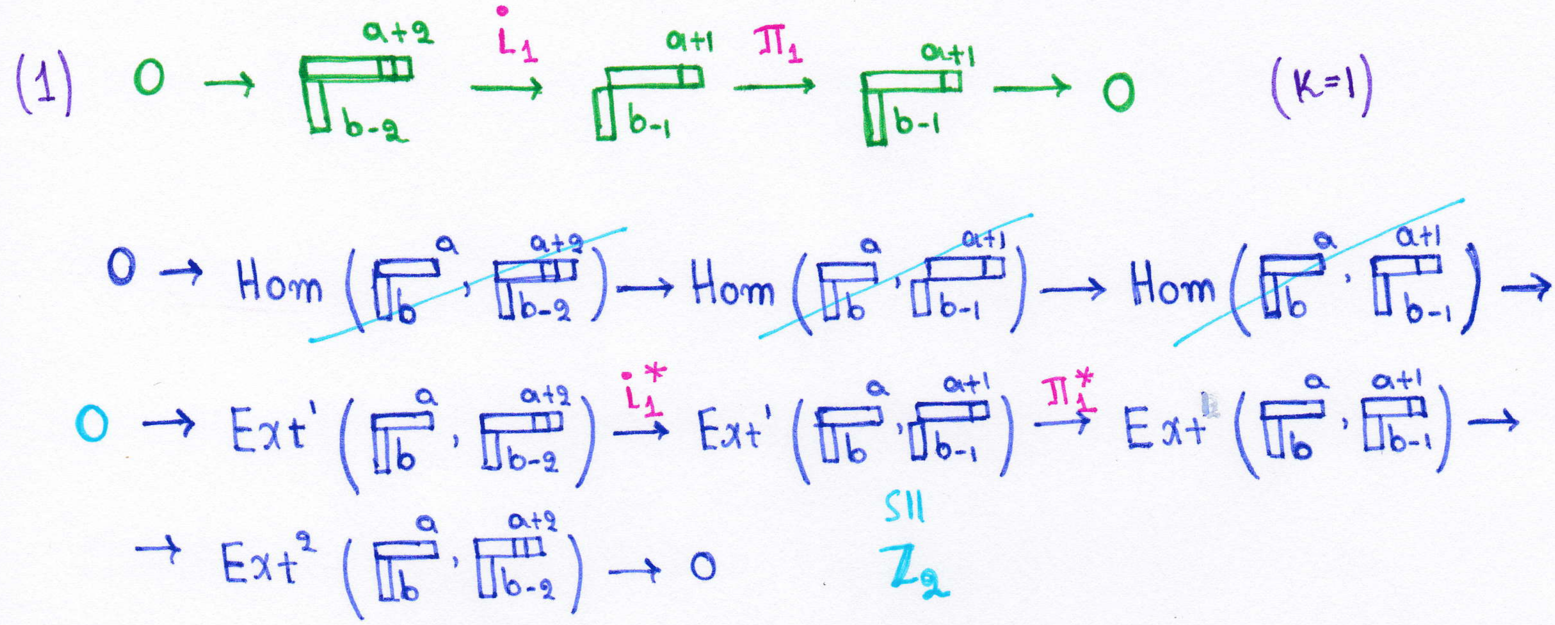


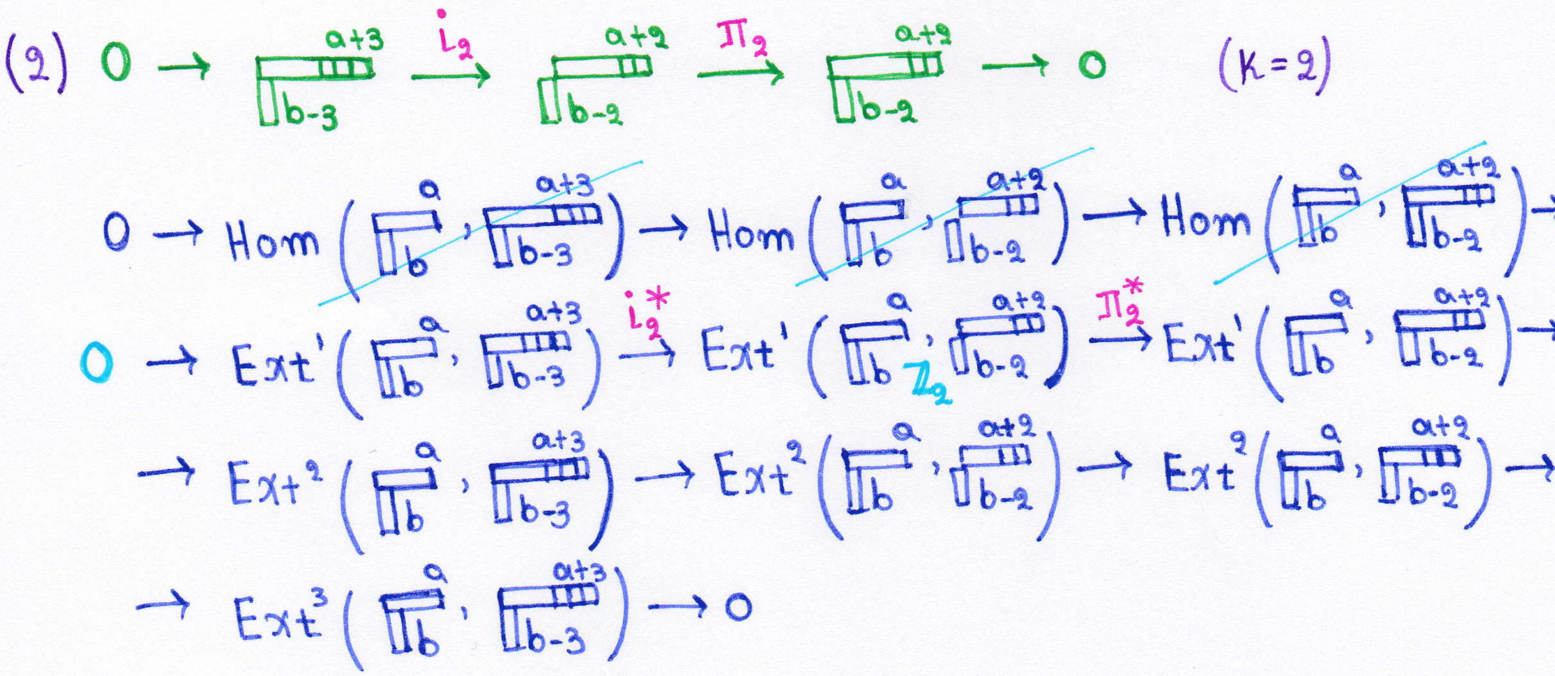
So,



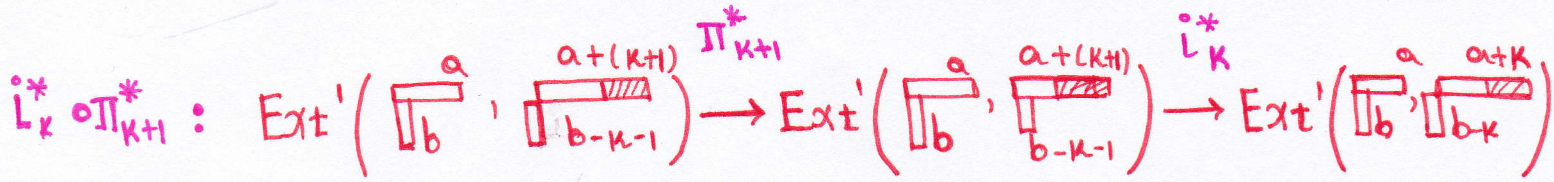
$\forall\ k \leq b$

For example :



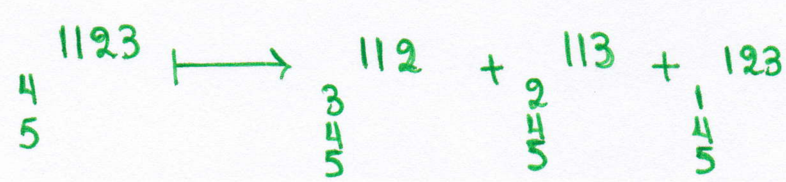


i_K^* : injective $\Rightarrow$ composition map

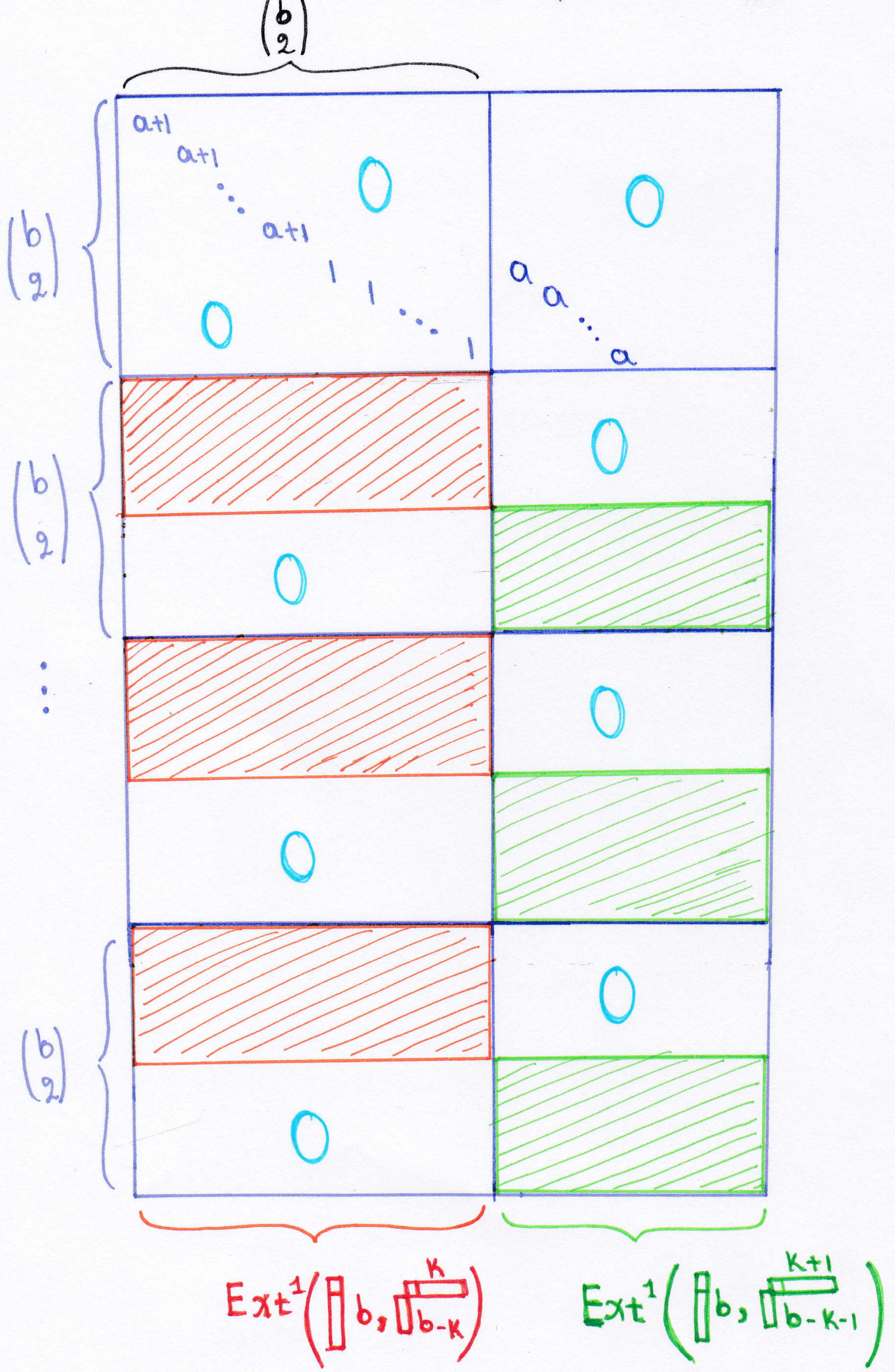


given by the diagonalization of one row - element and multiplicate this with the column of the skew hook.

examle :



• Matrices for Ext^1



$a=1,\ b'=b-1,\ k'=k-1$

$a=1,\ b'=b-1,\ k'=k$

- The matrices form is a consequence of the projective resolutions

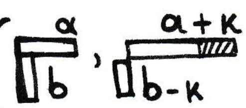
- Using induction on K (= number of boxes we move from the column to the row) we see that every row in our matrices sums to $0 \bmod 2$

- Let C_i, $i=1,\dots,\binom{b+1}{3}$ the matrix columns, then

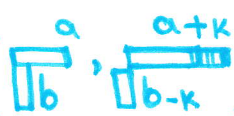
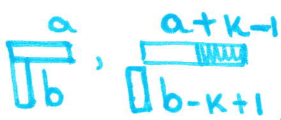
$$a\cdot\left(C_1+\dots+C_{\binom{b}{2}}\right) + \left(C_{\binom{b}{2}+1}+\dots+C_{\binom{b+1}{2}}\right) \equiv 0 \bmod 2$$

and

$$\frac{1}{2}\left[a\cdot\left(C_1+\dots+C_{\binom{b}{2}}\right) + \left(C_{\binom{b}{2}+1}+\dots+C_{\binom{b+1}{2}}\right)\right] \neq 0$$

is a generator for $\mathrm{Ext}^1\Big($  $\Big)$.

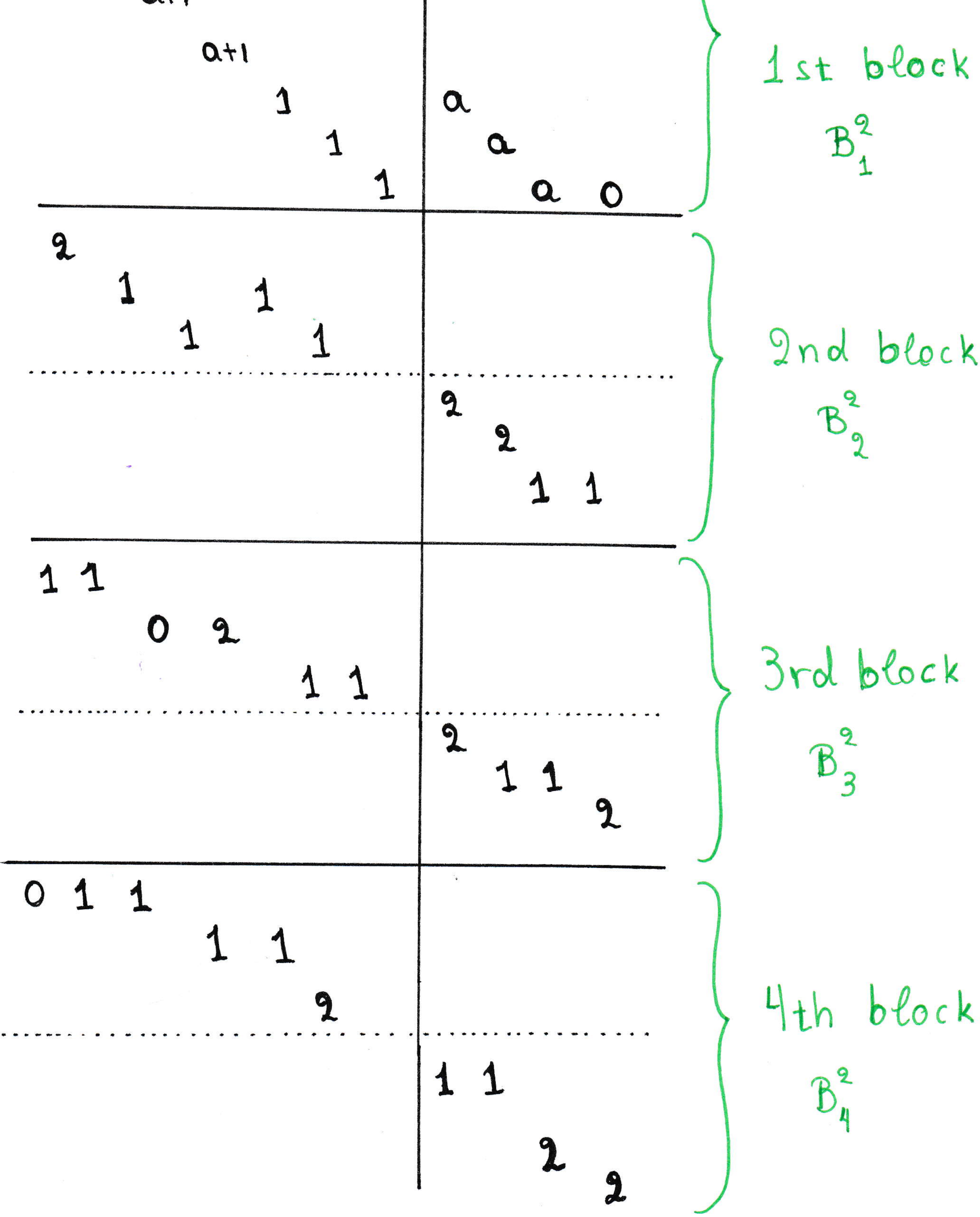
Now we calculate the image of this generator, let Γ_K, under the map $i^*_{K-1}\circ \pi^*_K$ wich sends

$\mathrm{Ext}^1\Big($  $\Big) \longrightarrow \mathrm{Ext}^1\Big($  $\Big)$

Example : $k=2$ and $b=4$

$\binom{b+1}{3} = \binom{5}{3} = 10$ columns

$b \cdot \binom{b}{2} = 4 \cdot \binom{4}{2} = 24$ rows



Example : $k=1$ and $b=4$

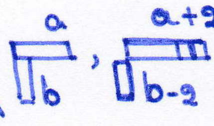
$$
\left.\begin{array}{cccc|ccc|cc|c}
a+1 & & & & & & & & & \\
 & 1 & & & a & & & & & \\
 & & 1 & & & a & & & & \\
 & & & 1 & & & a & & & \\
\hline
1 & 1 & & & & & & & & \\
 & & & & 2 & & & & & \\
 & & & & & 1 & & 1 & & \\
 & & & & & & 1 & & 1 & \\
\hline
 & 1 & 1 & & & & & & & \\
 & & & & 1 & 1 & & & & \\
 & & & & & & & 2 & & \\
 & & & & & & & & 1 & 1 \\
\hline
 & & 1 & 1 & & & & & & \\
 & & & & & 1 & 1 & & & \\
 & & & & & & & 1 & 1 & \\
 & & & & & & & & & 2
\end{array}\right.
$$

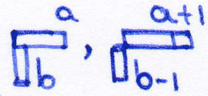
1st block B_1^1

2nd block B_2^1

3rd block B_3^1

4th block B_4^1

$\{T_i\}$ basis for $\mathrm{Ext}^1\big($  $\big)$

$\{S_j\}$ basis for $\mathrm{Ext}^1\big($  $\big)$

under $i^*_{k-1} \circ \pi^*_k$ we have

(here $k=2$)

$T_1 \longmapsto S_1 + S_2$

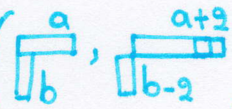
$T_2 \longmapsto -S_1 + S_3$

$T_3 \longmapsto S_1 + S_4$

$T_4 \longmapsto -S_2 - S_3$

$T_5 \longmapsto S_2 - S_4$

$T_6 \longmapsto S_3 + S_4$

for the 1st block of $\mathrm{Ext}^1\big($  $\big)$

$T_7 \longmapsto S_5 + S_6$

$T_8 \longmapsto -S_5 + S_7$

$T_9 \longmapsto S_5 + S_8$

$T_{10} \longmapsto -S_6 - S_7$

$T_{11} \longmapsto S_6 - S_8$

$T_{12} \longmapsto S_7 + S_8$

for the 2nd block

. . . .

$$\Gamma_2 = \frac{1}{2} a \left(C_1^2 + C_2^2 + \dots + C_{\binom{b}{2}}^2 \right) + \frac{1}{2} \left(C_{\binom{b}{2}+1}^2 + \dots + C_{\binom{b+1}{3}}^2 \right)$$

(for b=4)

$$= \frac{1}{2} a \left(C_1^2 + C_2^2 + \dots + C_6^2 \right) + \frac{1}{2} \left(C_7^2 + \dots + C_{10}^2 \right)$$

$$= \binom{a+1}{2} \cdot (T_1 + T_2 + T_3) + a \cdot (T_4 + T_5 + T_6) \quad \text{1st block}$$

$$+ a \cdot (T_7 + T_8 + T_9) + (T_{10} + T_{11} + T_{12}) \quad \text{2nd block}$$

$$+ a \cdot (T_{13} + T_{14} + T_{15}) + (T_{16} + T_{17} + T_{18}) \quad \text{3rd block}$$

$$+ a \cdot (T_{19} + T_{20} + T_{21}) + (T_{22} + T_{23} + T_{24}) \quad \text{4th block}$$

$$i_1^* \circ \pi_2^* (\Gamma_2) = \binom{a+1}{2} \cdot (S_1 + S_2 + S_3 + S_4) + \frac{1}{2} a (S_5 + \dots + S_{16})$$

$$\Gamma_1 = \frac{1}{2} a \left(C_1^1 + C_2^1 + \dots + C_b^1 \right) + \frac{1}{2} \left(C_{b+1}^1 + \dots + C_{\binom{b+1}{3}}^1 \right)$$

(for b=4)

$$= \binom{a+1}{2} S_1 + a \cdot (S_2 + S_3 + S_4) \quad \text{1st block}$$

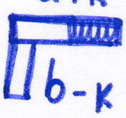
$$+ a \cdot S_5 + (S_6 + S_7 + S_8) \quad \text{2nd block}$$

$$+ a \cdot S_9 + (S_{10} + S_{11} + S_{12}) \quad \text{3rd block}$$

$$+ a \cdot S_{13} + (S_{14} + S_{15} + S_{16}) \quad \text{4th block}$$

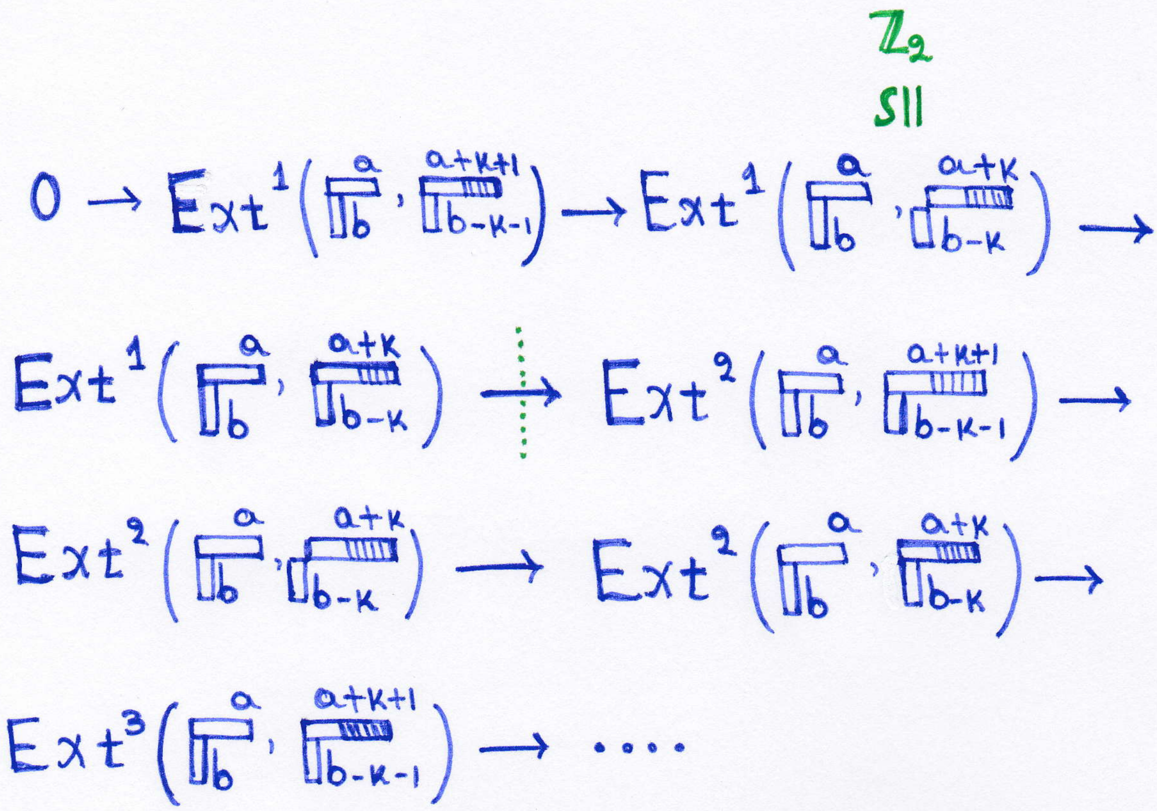
and $i_1^* \circ \pi_2^*(\Gamma_2) - a \cdot \Gamma_1 = -\binom{a}{2}\left(c_1^1 + c_2^1 + c_3^1 + c_4^1\right)$
$= 0$

Theorem:

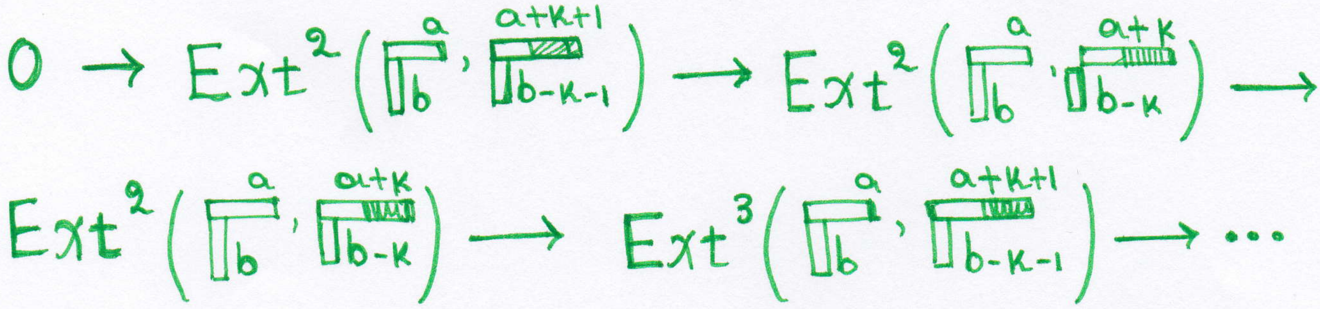
$\mathrm{Ext}^1\Big($  $^{a}_{b}$,  $^{a+k}_{b-k}\Big) = \begin{cases} \mathbb{Z}_{a+b} & k=1 \\ \dfrac{\mathbb{Z}_2}{\mathrm{hcf}(2, a+b-k)} & k \geq 2 \end{cases}$

The case $k=1$ is solved

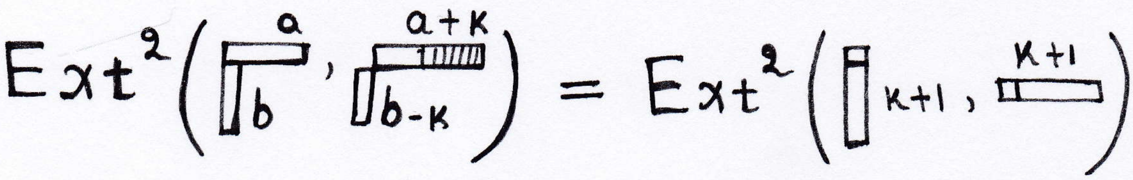
<u>see</u>: Maliakas, Resolutions, homological dimensions and extensions of hook representations, Commun. Algebra, 19, (1991) (2195 - 2216)



Using our previous theorem we have:



where by Kulkarni we have that



and $\operatorname{Ext}^2(\Lambda^{k+1}, D^{k+1}) = 0$ for $k > 4$

(needs calculations)

So, $\mathrm{Ext}^2\left(\Gamma^{a}_{b},\ \Gamma^{a+k}_{b-k}\right) = 0 \quad \forall\ k > 4$

What happens when $k = 2, 3$ or 4?

$$\mathbb{Z}_3 \cong \mathrm{Ext}^2\left(\Gamma^{a}_{b},\ \Gamma^{a+2}_{b-2}\right)$$

$$0 \to \mathrm{Ext}^2\left(\Gamma^{a}_{b},\ \Gamma^{a+3}_{b-3}\right) \xrightarrow{i_3^*} \mathrm{Ext}^2\left(\Gamma^{a}_{b},\ \Gamma^{a+2}_{b-2}\right) \xrightarrow{\pi_2^*}$$

$$\mathrm{Ext}^2\left(\Gamma^{a}_{b},\ \Gamma^{a+2}_{b-2}\right) \to \mathrm{Ext}^3\left(\Gamma^{a}_{b},\ \Gamma^{a+3}_{b-3}\right) \to 0$$

By Kulkarni we have:

$$\mathrm{Ext}^2\left(\Gamma^{a}_{b},\ \Gamma^{a+2}_{b-2}\right) = \mathrm{Ext}^2\left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array},\ \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}\right) = \mathbb{Z}_3$$

so $\mathrm{Ext}^2\left(\Gamma^{a}_{b},\ \Gamma^{a+3}_{b-3}\right) = 0$ or $\mathbb{Z}_3$

We also know that :

$$\mathrm{Ext}^2\left(\Gamma^{a}_{b}, \Gamma^{a+2}_{b-2}\right) = \begin{cases} a+b, & \text{when } a+b \equiv 1 \bmod 2 \\ \frac{a+b}{2}, & \text{when } a+b \equiv 0 \bmod 2 \end{cases}$$

because of the following exact sequence :

$$0 \to \mathrm{Ext}^1\left(\Gamma^{a}_{b}, \Gamma^{a+2}_{b-2}\right) \longrightarrow \underset{\mathbb{Z}_2}{\underset{\cong}{\mathrm{Ext}^1\left(\Gamma^{a}_{b}, \Gamma^{a+1}_{b-1}\right)}} \longrightarrow \underset{\mathbb{Z}_{a+b}}{\underset{\cong}{\mathrm{Ext}^1\left(\Gamma^{a}_{b}, \Gamma^{a+1}_{b-1}\right)}}$$

$$\longrightarrow \mathrm{Ext}^2\left(\Gamma^{a}_{b}, \Gamma^{a+2}_{b-2}\right) \longrightarrow 0$$

so we get an exact sequence of the form :

$$0 \to \mathrm{Ext}^2\left(\Gamma^{a}_{b}, \Gamma^{a+3}_{b-3}\right) \xrightarrow{i_3^*} \mathbb{Z}_3 \xrightarrow{\pi_2^*} \mathbb{Z}_{a+b} \text{ or } \mathbb{Z}_{\frac{a+b}{2}} \longrightarrow$$

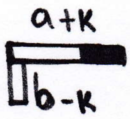
$$\mathrm{Ext}^3\left(\Gamma^{a}_{b}, \Gamma^{a+3}_{b-3}\right) \longrightarrow 0$$

If $3 \nmid a+b \Leftrightarrow \pi_2^* \equiv 0 \Leftrightarrow$

$$\mathrm{Ext}^2\left(\Gamma^{a}_{b}, \Gamma^{a+3}_{b-3}\right) = \mathbb{Z}_3$$

Theorem :

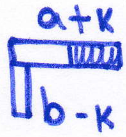
$$Ext^2\left(\right.$$ 

, 

$$\left.\right) = \begin{cases} 0 & k = 1 \\ \mathbb{Z}_{\frac{a+b}{hcf(2,a+b)}} & k = 2 \\ \mathbb{Z}_{\frac{3}{hcf(3,a+b)}} & k = 3 \\ \mathbb{Z}_{hcf(3,a+b)} & k = 4 \\ 0 & k \geq 5 \end{cases}$$

Theorem:

Ext^{k} (

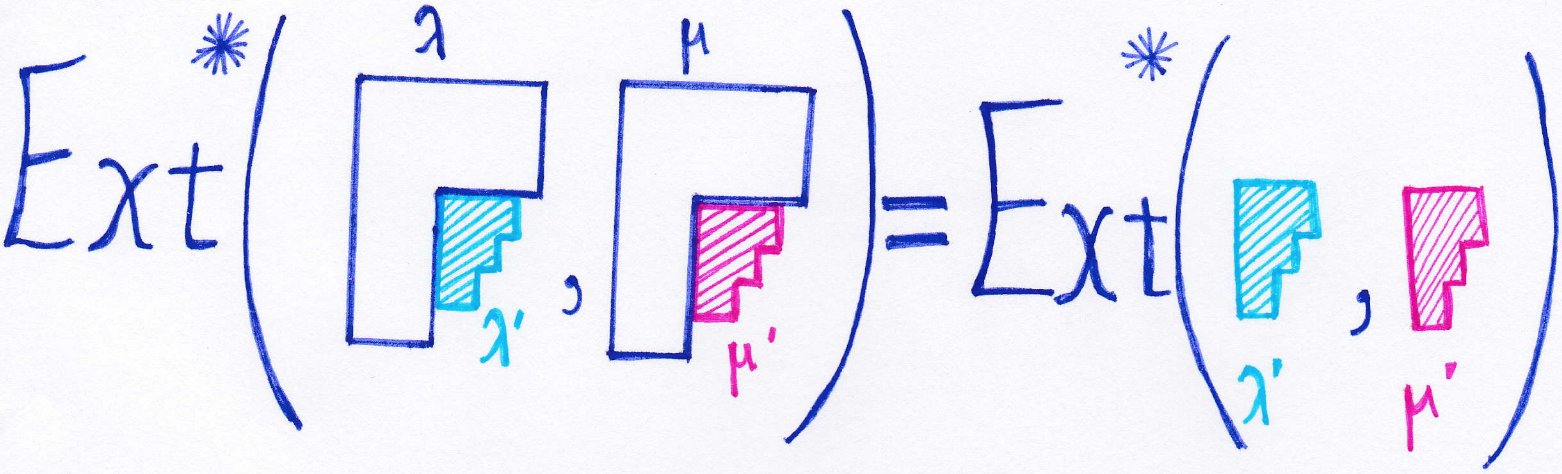
, 

) $\cong \mathbb{Z}_{d_k}$

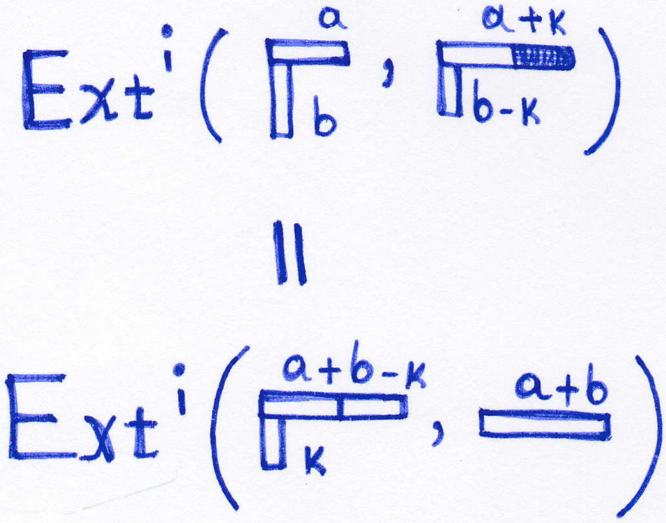
where

$$d_k = \mathrm{hcf}\left(\binom{a+b}{1}, \binom{a+b}{2}, \ldots, \binom{a+b}{k}\right)$$

• **Remark:** "Column and row removals"



• **Conjecture:**



depends only on $a+b$ and k