

On Extensions
of Hook Weyl modules

Stergiopoulou
Dimitra - Dionysia

University of Athens

Polynomial Representation of GL_n

Notation: $\lambda = \text{partition}$

$\Delta(\lambda) = \text{Weyl module}$

$\nabla(\lambda) = \text{Costandard (or Schur) module}$

$L(\lambda) = \text{Simple module}$

Main open problem:

The determination of $\text{ch}L(\lambda)$

$M_{\mathbb{K}}(n, r) :=$ finite-dimensional
homogeneous of degree r polynomial
representations of $GL_n(\mathbb{K})$

($\mathbb{K} = \text{field or } \mathbb{Z}$)

The problem is solved in some special cases:

- λ, μ have at most two parts, $i=1$
 - ~ **Akin** and **Buchsbaum**, Characteristic-free representation theory of the general linear group II: Homological considerations, Adv. in Math. 72 (1988), 171-210. **Integral**
 - ~ **Buchsbaum**, Aspects of characteristic-free representation theory of GL_n and some applications to intertwining numbers, Acta Appl. Math. 21 (1990), 247-261. **Integral**
 - ~ **Erdmann**, Ext' for Weyl modules of $SL_2(K)$, Mathematische Zeitschrift 2 (1994), 447-459. **Modular**
- λ, μ are hook partitions, $\mu - \lambda = \text{positive root}$, $i=1$
 - ~ **Maliakas**, Resolutions, homological dimensions and extensions of hook representations, Commun. Algebra, 19 (1991), 2195-2216 **Integral**
- λ, μ are ρ -hook partitions for every i
 - ~ **Doty** and **Martin**, Hook modules for general linear groups, Arch. Math 92 (2009), 206-214 **Modular**
- $\mu - \lambda = \text{positive root}$, $i=1$
 - ~ **Kulkarni**, On the Ext groups for Weyl modules of GL_n , J. Algebra 304 (2006), 510-542 **Integral**
- λ, μ have at most 3 parts, $i=1$
 - ~ **Buchsbaum** and **Flores**, Intertwining numbers: the three rowed case, J. Algebra 183 (1996), 605-635 **Integral**

- λ, μ have at most 2 parts, $\forall i$
 - ~ Parker, Higher extensions between modules for SL_2 , Adv. Math. 209 (2007) no.1, 381-405 Modular
 - λ, μ have at most 3 parts and $i=0$
 - ~ Cox and Parker, Homomorphisms between Weyl modules for SL_3 , Transaction AMS, 358, 4159-4207 Modular
-

It is known that

$$\text{ch} L(\lambda) = \sum (-1)^i \dim \text{Ext}^i(L(\lambda), \Delta(\mu)) \cdot s_\mu$$

where $s_\mu = \text{ch} \Delta(\mu)$ is the Schur function
and the Ext groups are taken over $M_{\mathbb{K}}(n, r)$
 \mathbb{K} is a field,

and

$$\text{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \text{Ext}^i(L(\nu), \Delta(\mu)) \neq 0$$

where $L(\nu)$ is quotient in a composition series of $\Delta(\lambda)$.

So, we are interested in the
determination of $\text{Ext}^i(\Delta(\lambda), \Delta(\mu))$

Why Hooks?

Hook Weyl modules appear as duals of the cycles in the Koszul complex:

$$0 \rightarrow \dots \rightarrow \Lambda^{a+1} V \otimes S_{b-1} V \rightarrow \Lambda^a V \otimes S_b V \rightarrow \Lambda^{a-1} V \otimes S_{b+1} V \rightarrow \dots \rightarrow 0$$

V = the natural GL_n -module

General fact:

$$\text{If } \text{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0$$

$\Rightarrow \mu > \lambda$ in dominance order

So here we examine the case where

$$\lambda = \begin{array}{|c|} \hline a \\ \hline \square \\ \hline b \\ \hline \end{array} \quad \text{and} \quad \mu = \begin{array}{|c|} \hline a+k \\ \hline \square \\ \hline b-k \\ \hline \end{array}$$

We will consider the corresponding integral Ext-groups $(GL_n(\mathbb{Z}))$.

We then obtain the modular Ext-groups via the

"Universal Coefficient Theorem"

Theorem:

$$\text{Ext}^1\left(\prod_b^a, \prod_{b-k}^{a+k}\right) = \begin{cases} \mathbb{Z}_{a+b} & k=1 \\ \frac{\mathbb{Z}_2}{\text{hcf}(2, a+b-k)} & k \geq 2 \end{cases}$$

The case $k=1$ is solved (Kulkarni)

Theorem :

$$\text{Ext}^2 \left(\begin{array}{c} a \\ \square \\ b \end{array}, \begin{array}{c} a+k \\ \square \\ b-k \end{array} \right) = \begin{cases} 0 & k=1 \\ \mathbb{Z} \frac{a+b}{\text{hcf}(2, a+b)} & k=2 \\ \mathbb{Z} \frac{3}{\text{hcf}(3, a+b)} & k=3 \\ \mathbb{Z} \text{hcf}(3, a+b) & k=4 \\ 0 & k \geq 5 \end{cases}$$

Theorem :

$$\text{Ext}^k \left(\begin{array}{c} a \\ \square \\ b \end{array}, \begin{array}{c} a+k \\ \square \\ b-k \end{array} \right) \cong \mathbb{Z}_{d_k}$$

where

$$d_k = \text{hcf} \left(\binom{a+b}{1}, \binom{a+b}{2}, \dots, \binom{a+b}{k} \right)$$

• Our tools:

1) Projective resolutions for hook Weyl modules over the integral Schur algebra $S_{\mathbb{Z}}(n, r)$

(Maliakas, Resolutions, homological dimensions and extensions of hook representations, *Commun. Algebra* 19, (1991), 2195-2216)

2) Long exact sequence obtained from

$$0 \rightarrow \Delta(a+1, 1^{b-1}) \rightarrow \Lambda^b V \otimes D_a V \rightarrow \Delta(a, 1^b) \rightarrow 0$$

(where $DV =$ the divided power algebra of V)

and the functor $\text{Hom}_{S_{\mathbb{Z}}}(\Delta(\lambda), -)$

$$0 \rightarrow \begin{array}{|c|} \hline a+1 \\ \hline \hline \\ \hline b-1 \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline a \\ \hline \hline \\ \hline b \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline a \\ \hline \hline \\ \hline b \\ \hline \end{array} \rightarrow 0$$

3) $\text{Ext}_{S_Z}^i(\Delta(\lambda), \Lambda^q V \otimes D_b V) = \text{Ext}_{S_Z}^i(\Lambda^q V, D_q V)$

for appropriate $q \in \mathbb{N}^*$

(Kulkarni, Skew Weyl modules for GL_n and degree reduction for Schur algebras, J. Algebra 224 (2000), 248-262.)

4) Computations with weight spaces of Weyl modules

Kulkarni : $\text{Ext}^* \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}, \begin{array}{|c|} \hline a+k \\ \hline b-k+1 \\ \hline \end{array} \right) = \text{Ext}^* \left(\begin{array}{|c|} \hline a-1 \\ \hline k \\ \hline \end{array}, \begin{array}{|c|} \hline a+k-1 \\ \hline \\ \hline \end{array} \right)$

corollary

A-B $\text{Ext}^* \left(\begin{array}{|c|} \hline a+k-1 \\ \hline \\ \hline \end{array}, \begin{array}{|c|} \hline k \\ \hline a-1 \\ \hline \end{array} \right)$

Kulkarni
again

$\text{Ext}^* \left(\begin{array}{|c|} \hline k \\ \hline \\ \hline \end{array}, \begin{array}{|c|} \hline k \\ \hline \\ \hline \end{array} \right)$

and here
 $a=1, b=k-1$

When $*=1 \Rightarrow \text{Ext}^1 \left(\begin{array}{|c|} \hline k \\ \hline \\ \hline \end{array}, \begin{array}{|c|} \hline k \\ \hline \\ \hline \end{array} \right) \cong \mathbb{Z}_2$

or equivalently $\text{Ext}^1 \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}, \begin{array}{|c|} \hline a+k-1 \\ \hline b-k+1 \\ \hline \end{array} \right) \cong \mathbb{Z}_2$

$0 \rightarrow \begin{array}{|c|} \hline a+k \\ \hline b-k \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline a+k-1 \\ \hline b-k+1 \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline a+k-1 \\ \hline b-k+1 \\ \hline \end{array} \rightarrow 0$

$0 \rightarrow \text{Hom} \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}, \begin{array}{|c|} \hline a+k \\ \hline b-k \\ \hline \end{array} \right) \rightarrow \text{Hom} \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}, \begin{array}{|c|} \hline a+k-1 \\ \hline b-k+1 \\ \hline \end{array} \right) \rightarrow \text{Hom} \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}, \begin{array}{|c|} \hline a+k-1 \\ \hline b-k+1 \\ \hline \end{array} \right) \rightarrow$

$0 \rightarrow \text{Ext}^1 \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}, \begin{array}{|c|} \hline a+k \\ \hline b-k \\ \hline \end{array} \right) \rightarrow \text{Ext}^1 \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}, \begin{array}{|c|} \hline a+k-1 \\ \hline b-k+1 \\ \hline \end{array} \right) \rightarrow \text{Ext}^1 \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}, \begin{array}{|c|} \hline a+k-1 \\ \hline b-k+1 \\ \hline \end{array} \right) \rightarrow$

...
 \mathbb{Z}_2

So,

$\text{Ext}^1 \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}, \begin{array}{|c|} \hline a+k \\ \hline b-k \\ \hline \end{array} \right) \cong \mathbb{Z}_2 \text{ or } 0 \quad \forall k \leq b$

• Matrices for $\text{Ext}^1\left(\begin{smallmatrix} a \\ \mathbb{Z}_b \end{smallmatrix}, \begin{smallmatrix} a+k \\ \mathbb{Z}_{b-k} \end{smallmatrix}\right)$

	$\left(\begin{smallmatrix} b \\ 2 \end{smallmatrix}\right)$	
$\left(\begin{smallmatrix} b \\ 2 \end{smallmatrix}\right)$	$a+1$ $a+1$ \dots $a+1$ 0	0 0 \dots 0 0
	1 1 \dots 1 1	a a \dots a a
$\left(\begin{smallmatrix} b \\ 2 \end{smallmatrix}\right)$		0
⋮		0
		0
		0
$\left(\begin{smallmatrix} b \\ 2 \end{smallmatrix}\right)$		0

$$\text{Ext}^1\left(\begin{smallmatrix} k \\ \mathbb{Z}_b \end{smallmatrix}, \begin{smallmatrix} k \\ \mathbb{Z}_{b-k} \end{smallmatrix}\right)$$

$$a=1, b'=b-1, k'=k-1$$

$$\text{Ext}^1\left(\begin{smallmatrix} k+1 \\ \mathbb{Z}_b \end{smallmatrix}, \begin{smallmatrix} k+1 \\ \mathbb{Z}_{b-k-1} \end{smallmatrix}\right)$$

$$a=1, b'=b-1, k'=k$$

- The matrices form is a consequence of the projective resolutions
- Using induction on k (= number of boxes we move from the column to the row) we see that every row in our matrices sums to $0 \pmod 2$
- Let $C_i, i=1, \dots, \binom{b+1}{2}$ the matrix columns, then

$$a \cdot (C_1 + \dots + C_{\binom{b}{2}}) + (C_{\binom{b}{2}+1} + \dots + C_{\binom{b+1}{2}}) \equiv 0 \pmod 2$$

and

$$\frac{1}{2} \left[a \cdot (C_1 + \dots + C_{\binom{b}{2}}) + (C_{\binom{b}{2}+1} + \dots + C_{\binom{b+1}{2}}) \right] \neq 0$$

is a generator for $\text{Ext}^1 \left(\overline{\mathbb{F}}_b^a, \overline{\mathbb{F}}_{b-k}^{a+k} \right)$.

Now we calculate the image of this generator, let Γ_k , under the map $i_{k-1}^* \circ \Pi_k^*$ which sends

$$\text{Ext}^1 \left(\overline{\mathbb{F}}_b^a, \overline{\mathbb{F}}_{b-k}^{a+k} \right) \longrightarrow \text{Ext}^1 \left(\overline{\mathbb{F}}_b^a, \overline{\mathbb{F}}_{b-k+1}^{a+k-1} \right)$$

Example: $k=2$ and $b=4$

$$\binom{b+1}{3} = \binom{5}{3} = 10 \text{ columns}$$

$$b \cdot \binom{b}{2} = 4 \cdot \binom{4}{2} = 24 \text{ rows}$$

$a+1$									
	$a+1$					a			
		$a+1$					a		
			1					a	
				1					0
2									
	1								
		1							
			1						
				1					
						2			
							2		
								1	1
1	1								
		0	2						
				1					
						2			
							1	1	
									2
0	1	1							
			1	1					
				2					
						1	1		
								2	
									2

1st block

$$B_1^2$$

2nd block

$$B_2^2$$

3rd block

$$B_3^2$$

4th block

$$B_4^2$$

Example : $k=1$ and $b=4$

$a+1$				} 1st block B_1^1	
1		a			
	1	a			
	1		a	} 2nd block B_2^1	
1	1				
		2	1		
			1	1	
	1	1		} 3rd block B_3^1	
			2		
			1		1
		1	1	} 4th block B_4^1	
	1	1			
			1		1
				2	

$\{T_i\}$ basis for $\text{Ext}^1\left(\begin{smallmatrix} a \\ \square_b \end{smallmatrix}, \begin{smallmatrix} a+2 \\ \square_{b-2} \end{smallmatrix}\right)$

$\{S_j\}$ basis for $\text{Ext}^1\left(\begin{smallmatrix} a \\ \square_b \end{smallmatrix}, \begin{smallmatrix} a+1 \\ \square_{b-1} \end{smallmatrix}\right)$

under $i_{k-1}^* \circ \pi_k^*$ we have

$$T_1 \mapsto S_1 + S_2$$

$$T_2 \mapsto -S_1 + S_3$$

$$T_3 \mapsto S_1 + S_4$$

$$T_4 \mapsto -S_2 - S_3$$

$$T_5 \mapsto S_2 - S_4$$

$$T_6 \mapsto S_3 + S_4$$

(here $k=2$)

for the 1st block

of $\text{Ext}^1\left(\begin{smallmatrix} a \\ \square_b \end{smallmatrix}, \begin{smallmatrix} a+2 \\ \square_{b-2} \end{smallmatrix}\right)$

$$T_7 \mapsto S_5 + S_6$$

$$T_8 \mapsto -S_5 + S_7$$

$$T_9 \mapsto S_5 + S_8$$

$$T_{10} \mapsto -S_6 - S_7$$

$$T_{11} \mapsto S_6 - S_8$$

$$T_{12} \mapsto S_7 + S_8$$

for the 2nd block

• • • •

$$\Gamma_2 = \frac{1}{2} a (C_1^2 + C_2^2 + \dots + C_{\binom{b}{2}}^2) + \frac{1}{2} (C_{\binom{b}{2}+1}^2 + \dots + C_{\binom{b+1}{3}}^2)$$

(for $b=4$)

$$\underline{\underline{=}} \frac{1}{2} a (C_1^2 + C_2^2 + \dots + C_6^2) + \frac{1}{2} (C_7^2 + \dots + C_{10}^2)$$

$$\underline{\underline{=}} \binom{a+1}{2} \cdot (T_1 + T_2 + T_3) + a \cdot (T_4 + T_5 + T_6) \quad \text{1st block}$$

$$+ a \cdot (T_7 + T_8 + T_9) + (T_{10} + T_{11} + T_{12}) \quad \text{2nd block}$$

$$+ a \cdot (T_{13} + T_{14} + T_{15}) + (T_{16} + T_{17} + T_{18}) \quad \text{3rd block}$$

$$+ a \cdot (T_{19} + T_{20} + T_{21}) + (T_{22} + T_{23} + T_{24}) \quad \text{4th block}$$

$$i_1^* \circ \Pi_2^* (\Gamma_2) = \binom{a+1}{2} \cdot (S_1 + S_2 + S_3 + S_4) + \frac{1}{2} a (S_5 + \dots + S_{16})$$

$$\Gamma_1 = \frac{1}{2} a (C_1^1 + C_2^1 + \dots + C_b^1) + \frac{1}{2} (C_{b+1}^1 + \dots + C_{\binom{b+1}{3}}^1)$$

(for $b=4$)

$$\binom{a+1}{2} S_1 + a \cdot (S_2 + S_3 + S_4) \quad \text{1st block}$$

$$+ a \cdot S_5 + (S_6 + S_7 + S_8) \quad \text{2nd block}$$

$$+ a \cdot S_9 + (S_{10} + S_{11} + S_{12}) \quad \text{3rd block}$$

$$+ a \cdot S_{13} + (S_{14} + S_{15} + S_{16}) \quad \text{4th block}$$

$$\text{and } i_1^* \cdot \pi_2^* (\Gamma_2) - a \cdot \Gamma_1 = -\binom{a}{2} (C_1^1 + C_2^1 + C_3^1 + C_4^1) = 0$$

Theorem:

$$\text{Ext}^1 \left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k \\ \hline b-k \end{array} \right) = \begin{cases} \mathbb{Z}_{a+b} & k=1 \\ \frac{\mathbb{Z}_2}{\text{hcf}(2, a+b-k)} & k \geq 2 \end{cases}$$

The case $k=1$ is solved

see : Maliakas, Resolutions, homological dimensions and extensions of hook representations, Commun. Algebra, 19, (1991) (2195 - 2216)

\mathbb{Z}_2

SII

$$\begin{aligned}
0 &\rightarrow \text{Ext}^1\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k+1 \\ \hline b-k-1 \end{array}\right) \rightarrow \text{Ext}^1\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k \\ \hline b-k \end{array}\right) \rightarrow \\
&\text{Ext}^1\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k \\ \hline b-k \end{array}\right) \xrightarrow{\dots} \text{Ext}^2\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k+1 \\ \hline b-k-1 \end{array}\right) \rightarrow \\
&\text{Ext}^2\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k \\ \hline b-k \end{array}\right) \rightarrow \text{Ext}^2\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k \\ \hline b-k \end{array}\right) \rightarrow \\
&\text{Ext}^3\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k+1 \\ \hline b-k-1 \end{array}\right) \rightarrow \dots
\end{aligned}$$

Using our previous theorem we have:

$$\begin{aligned}
0 &\rightarrow \text{Ext}^2\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k+1 \\ \hline b-k-1 \end{array}\right) \rightarrow \text{Ext}^2\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k \\ \hline b-k \end{array}\right) \rightarrow \\
&\text{Ext}^2\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k \\ \hline b-k \end{array}\right) \rightarrow \text{Ext}^3\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k+1 \\ \hline b-k-1 \end{array}\right) \rightarrow \dots
\end{aligned}$$

where by Kulkarni we have that

$$\text{Ext}^2\left(\begin{array}{c} a \\ \hline b \end{array}, \begin{array}{c} a+k \\ \hline b-k \end{array}\right) = \text{Ext}^2\left(\begin{array}{c} \\ \hline k+1 \end{array}, \begin{array}{c} k+1 \\ \hline \end{array}\right)$$

and $\text{Ext}^2(\Lambda^{k+1}, D^{k+1}) = 0$ for $k > 4$

(needs calculations)

$$\text{So, } \text{Ext}^2 \left(\begin{array}{c} a \\ \square_b \end{array}, \begin{array}{c} a+k \\ \square_{b-k} \end{array} \right) = 0 \quad \forall k > 4$$

What happens when $k=2, 3$ or 4 ?

$$0 \rightarrow \text{Ext}^2 \left(\begin{array}{c} a \\ \square_b \end{array}, \begin{array}{c} a+3 \\ \square_{b-3} \end{array} \right) \xrightarrow{L_3^*} \text{Ext}^2 \left(\begin{array}{c} a \\ \square_b \end{array}, \begin{array}{c} a+2 \\ \square_{b-2} \end{array} \right) \xrightarrow{\Pi_2^*}$$

$S // \mathbb{Z}_3$

$$\text{Ext}^2 \left(\begin{array}{c} a \\ \square_b \end{array}, \begin{array}{c} a+2 \\ \square_{b-2} \end{array} \right) \rightarrow \text{Ext}^3 \left(\begin{array}{c} a \\ \square_b \end{array}, \begin{array}{c} a+3 \\ \square_{b-3} \end{array} \right) \rightarrow 0$$

By Kulkarni we have :

$$\text{Ext}^2 \left(\begin{array}{c} a \\ \square_b \end{array}, \begin{array}{c} a+2 \\ \square_{b-2} \end{array} \right) = \text{Ext}^2 \left(\begin{array}{c} \square \\ \square \end{array}, \begin{array}{c} \square \\ \square \end{array} \right) = \mathbb{Z}_3$$

$$\text{So } \text{Ext}^2 \left(\begin{array}{c} a \\ \square_b \end{array}, \begin{array}{c} a+3 \\ \square_{b-3} \end{array} \right) = 0 \text{ or } \mathbb{Z}_3$$

We also know that :

$$\text{Ext}^2 \left(\mathbb{Z}_b^a, \mathbb{Z}_{b-2}^{a+2} \right) = \begin{cases} a+b, & \text{when } a+b \equiv 1 \pmod{2} \\ \frac{a+b}{2}, & \text{when } a+b \equiv 0 \pmod{2} \end{cases}$$

because of the following exact sequence :

$$\begin{aligned} 0 \rightarrow \text{Ext}^1 \left(\mathbb{Z}_b^a, \mathbb{Z}_{b-2}^{a+2} \right) &\rightarrow \text{Ext}^1 \left(\mathbb{Z}_b^a, \mathbb{Z}_{b-1}^{a+1} \right) \rightarrow \text{Ext}^1 \left(\mathbb{Z}_b^a, \mathbb{Z}_{b-1}^{a+1} \right) \\ &\rightarrow \text{Ext}^2 \left(\mathbb{Z}_b^a, \mathbb{Z}_{b-2}^{a+2} \right) \rightarrow 0 \end{aligned}$$

$\begin{matrix} \text{SII} \\ \mathbb{Z}_2 \end{matrix}$
 $\begin{matrix} \text{SII} \\ \mathbb{Z}_{a+b} \end{matrix}$

so we get an exact sequence of the form :

$$\begin{aligned} 0 \rightarrow \text{Ext}^2 \left(\mathbb{Z}_b^a, \mathbb{Z}_{b-3}^{a+3} \right) &\xrightarrow{\cdot L_3^*} \mathbb{Z}_3 \xrightarrow{\pi_2^*} \mathbb{Z}_{a+b} \text{ or } \mathbb{Z}_{\frac{a+b}{2}} \rightarrow \\ \text{Ext}^3 \left(\mathbb{Z}_b^a, \mathbb{Z}_{b-3}^{a+3} \right) &\rightarrow 0 \end{aligned}$$

If $3 \nmid a+b \Leftrightarrow \pi_2^* \equiv 0 \Leftrightarrow$

$$\text{Ext}^2 \left(\mathbb{Z}_b^a, \mathbb{Z}_{b-3}^{a+3} \right) = \mathbb{Z}_3$$

Theorem :

$$\text{Ext}^2 \left(\sqrt[b]{a}, \sqrt[b-k]{a+k} \right) =$$

$$\begin{cases} 0 & k=1 \\ \mathbb{Z} \frac{a+b}{\text{hcf}(2, a+b)} & k=2 \\ \mathbb{Z} \frac{3}{\text{hcf}(3, a+b)} & k=3 \\ \mathbb{Z} \text{hcf}(3, a+b) & k=4 \\ 0 & k \geq 5 \end{cases}$$

Theorem :

$$\text{Ext}^k \left(\binom{a}{b}, \binom{a+k}{b-k} \right) \cong \mathbb{Z}_{d_k}$$

where

$$d_k = \text{hcf} \left(\binom{a+b}{1}, \binom{a+b}{2}, \dots, \binom{a+b}{k} \right)$$

- Remark: "Column and row removals"

$$\text{Ext}^* \left(\begin{array}{c} \lambda \\ \text{[Diagram: Young diagram with a blue shaded hook] } \\ \lambda' \end{array}, \begin{array}{c} \mu \\ \text{[Diagram: Young diagram with a red shaded hook] } \\ \mu' \end{array} \right) = \text{Ext}^* \left(\begin{array}{c} \text{[Diagram: Blue shaded hook] } \\ \lambda' \end{array}, \begin{array}{c} \text{[Diagram: Red shaded hook] } \\ \mu' \end{array} \right)$$

- Conjecture:

$$\text{Ext}^i \left(\begin{array}{c} a \\ \text{[Diagram: Hook with top bar length } a \text{ and bottom bar length } b \text{]} \\ b \end{array}, \begin{array}{c} a+k \\ \text{[Diagram: Hook with top bar length } a+k \text{ and bottom bar length } b-k \text{]} \\ b-k \end{array} \right)$$

depends only on
 $a+b$ and k

||

$$\text{Ext}^i \left(\begin{array}{c} a+b-k \\ \text{[Diagram: Hook with top bar length } a+b-k \text{ and bottom bar length } k \text{]} \\ k \end{array}, \begin{array}{c} a+b \\ \text{[Diagram: Hook with top bar length } a+b \text{ and bottom bar length } 0 \text{]} \\ 0 \end{array} \right)$$