FROM THE FRAMISATION OF THE TEMPERLEY–LIEB ALGEBRA TO THE JONES POLYNOMIAL: AN ALGEBRAIC APPROACH

MARIA CHLOUVERAKI

Abstract. We prove that the Framisation of the Temperley–Lieb algebra is isomorphic to a direct sum of matrix algebras over tensor products of classical Temperley–Lieb algebras. We use this result to obtain a closed combinatorial formula for the invariants for classical links obtained from a Markov trace on the Framisation of the Temperley–Lieb algebra. For a given link L, this formula involves the Jones polynomials of all sublinks of L, as well as linking numbers.

1. Introduction

The Temperley–Lieb algebra was introduced by Temperley and Lieb [TeLi] for its applications in statistical mechanics. Jones later showed that the Temperley–Lieb algebra can be seen as a quotient of the Iwahori–Hecke algebra of type A [Jo1, Jo2]. He defined a Markov trace on it, now known as the Jones–Ocneanu trace, and used it to construct his famous polynomial link invariant, the Jones polynomial. This trace is also obtained as a specialisation of a trace defined directly on the Iwahori–Hecke algebra of type A, which in turn yields another famous polynomial link invariant, the HOMFLYPT polynomial (also known as the 2-variable Jones polynomial) [HOMFLY, PT].

Yokonuma–Hecke algebras were introduced by Yokonuma [Yo] as generalisations of Iwahori–Hecke algebras. In particular, the Yokonuma–Hecke algebra of type A is the centraliser algebra associated to the permutation representation with respect to a maximal unipotent subgroup of the general linear group over a finite field. In later years, Juyumaya transformed its presentation to "almost" the one we use in this paper and defined a Markov trace on it [Ju1, JuKa, Ju2]. Following Jones's method, Juyumaya and Lambropoulou used this trace to construct invariants for framed [JuLa1, JuLa2], classical [JuLa3] and singular [JuLa4] links. The exact presentation for the Yokonuma–Hecke algebra used in this paper is due to the author and Poulain d'Andecy, who modified Juyumaya's generators in [ChPdA]. Although the construction of the Markov trace with the new generators remains similar, the invariants for framed and classical links obtained from it are not topologically equivalent to the Juyumaya–Lambropoulou ones. This was shown in [CJKL], where the new invariants were constructed and studied. From then on, these are the "standard" link invariants obtained from the Yokonuma–Hecke algebra of type A. As was shown in [CJKL], they are not topologically equivalent to the HOMFLYPT polynomial and they can be generalised to a 3-variable skein link invariant which is stronger than the HOMFLYPT. In the Appendix of [CJKL], Lickorish gave a closed combinatorial formula for the value of these invariants on a link L which involves the HOMFLYPT polynomials of all sublinks of L and linking numbers. The same formula was obtained independently by Poulain d'Andecy and Wagner [PdAWa] with a method that we will discuss at the end of the introduction.

However, even prior to these recent results, there has been algebraic and topological interest in finding the analogue of the Temperley–Lieb algebra in the Yokonuma–Hecke algebra context. On the one hand, it would be a quotient of the Yokonuma–Hecke algebra of type A such that the Markov trace on it would yield a link invariant more general (and now known to be stronger) than the Jones polynomial. On the other hand, it would be an example of the "framisation technique" proposed in [JuLa5], according to which known

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algebras producing invariants for classical links can be enhanced with extra generators to produce invariants for framed links; the foremost example is the Yokonuma–Hecke algebra of type A which can be seen as the "framisation" of the Iwahori–Hecke algebra of type A.

Goundaroulis, Juyumaya, Kontogeorgis and Lambropoulou defined and studied three quotients of the Yokonuma–Hecke algebra of type A as potential candidates [GJKL1, GJKL2]. The one with the biggest topological interest was named "Framisation of the Temperley–Lieb algebra" and it is the one that produces the suitable generalisation of the Jones polynomial. The claim that this algebra is the natural analogue of the Temperley–Lieb algebra in this context is backed up algebraically by our findings in [ChPo2, ChPo3], where we studied the representation theory of this algebra and we proved the isomorphism theorem that we present in the current article (we also studied similarly the other two candidates in [ChPo1, ChPo3]). This isomorphism theorem states that the Framisation of the Temperley–Lieb algebra is isomorphic to a direct sum of matrix algebras over tensor products of Temperley–Lieb algebras. This result makes the Framisation of the Temperley–Lieb algebra the ideal analogue of the Temperley–Lieb algebra in view of Lusztig's isomorphism theorem [Lu], later reproved by Jacon and Poulain d'Andecy [JaPdA], Espinoza and Ryom–Hansen [EsRy] and Rostam [Ro], that states that the Yokonuma–Hecke algebra of type A is isomorphic to a direct sum of matrix algebras over tensor products of Iwahori–Hecke algebras of type A. To prove our result we use the exposition by Jacon and Poulain d'Andecy, where the presentation of the Yokonuma–Hecke algebra of [ChPdA] is used. In fact, in the current article we do not use the modified presentation that we used in [ChPo2, ChPo3], but we reprove the results with the presentation of [ChPdA] in order to be with agreement with the most recent topologically oriented papers on the subject (for example, [CJKL], [GoLa], [PdA], etc.). Finally, our isomorphism theorem allows us to determine a basis for the Framisation of the Temperley–Lieb algebra.

In the second part of the paper, we discuss the Markov traces on the Temperley–Lieb algebra and its Framisation, and explain how we can use them to define invariants for classical links from the former and for framed and classical links from the latter. We give several definitions of the traces. First, for the Jones–Ocneanu trace, we give the original definition of [Jo2] of a trace that needs to be normalised and re-scaled to produce a link invariant, and another one which is already invariant under positive and negative stabilisation. As far as the Juyumaya trace is concerned, the original definition of [Ju2] is also of a trace that needs to be normalised and re-scaled to produce a link invariant (under certain conditions discussed in detail in §4.3), and its stabilised version appears as a particular case of the Markov traces defined and classified by Jacon and Poulain d'Andecy in [JaPdA]. Using these stabilised traces and the isomorphism theorem for the Yokonuma–Hecke algebra, Poulain d'Andecy and Wagner in [PdAWa] obtained closed formulas that connect the values of these traces on a link L with the values of the HOMFLYPT polynomials of all sublinks of L, as well as their linking numbers. For a certain choice of parameters (see [PdA, Remarks 5.4] for details), they obtain Lickorish's formula. Here, we consider stabilised Markov traces on the Framisation of the Temperley–Lieb algebra, and thanks to our isomorphism theorem, we obtain an analogue of this formula for the link invariants obtained in this case; for a given link L, this formula involves the Jones polynomials of all sublinks of L and linking numbers. This formula has been obtained independently in [GoLa] as a specialisation of Lickorish's formula.

2. The Temperley–Lieb algebra and its Framisation

In this section, we give the definition of the Temperley–Lieb algebra as a quotient of the Iwahori–Hecke algebra of type A given by Jones [Jo2], as well as the definition of the Framisation of the Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra of type A given by Goundaroulis–Juyumaya–Kontogeorgis–Lambropoulou [GJKL2]. From now on, let $n \in \mathbb{N}$, $d \in \mathbb{N}^*$, and let q be an indeterminate. Set $R := \mathbb{C}[q, q^{-1}]$.

2.1. The Iwahori–Hecke algebra $\mathcal{H}_n(q)$.

The *Iwahori–Hecke algebra of type A*, denoted by $\mathcal{H}_n(q)$, is an R-associative algebra generated by the elements

$$G_1, \ldots, G_{n-1}$$

subject to the following braid relations:

$$
\begin{array}{rcll}
G_iG_j & = & G_jG_i & \text{for all } i,j=1,\ldots,n-1 \text{ with } |i-j|>1,\\
G_iG_{i+1}G_i & = & G_{i+1}G_iG_{i+1} & \text{for all } i=1,\ldots,n-2,
\end{array} \tag{2.1}
$$

together with the quadratic relations:

$$
G_i^2 = 1+(q-q^{-1})G_i \qquad \text{for all } i=1,\ldots,n-1. \tag{2.2}
$$

Remark 2.1. If we specialise q to 1, the defining relations (2.1)–(2.2) become the defining relations for the symmetric group $\mathfrak{S}_n$. Thus, the algebra $\mathcal{H}_n(q)$ is a deformation of $\mathbb{C}[\mathfrak{S}_n]$, the group algebra of $\mathfrak{S}_n$ over $\mathbb{C}$.

Remark 2.2. The relations (2.1) are defining relations for the classical braid group B_n on n strands. Thus, the algebra $\mathcal{H}_n(q)$ arises naturally as a quotient of the braid group algebra $R[B_n]$ over the quadratic relations (2.2).

Let $w\in\mathfrak{S}_n$ and let $w=s_{i_1}s_{i_2}\ldots s_{i_r}$ be a reduced expression for w, where s_i denotes the transposition $(i,i{+}1)$. We define $\ell(w):=r$ to be the *length* of w. By Matsumoto's lemma, the element $G_w:=G_{i_1}G_{i_2}\ldots G_{i_r}$ is well defined. It is well-known that the set $\mathcal{B}_{\mathcal{H}_n(q)}:=\{G_w\}_{w\in\mathfrak{S}_n}$ forms a basis of $\mathcal{H}_n(q)$ over R, which is called the *standard basis*. One presentation of the standard basis is the following:

$$
\mathcal{B}_{\mathcal{H}_n(q)} = \left\{ (G_{i_1}G_{i_1-1}\ldots G_{i_1-k_1})\ldots(G_{i_p}G_{i_p-1}\ldots G_{i_p-k_p}) \;\middle|\; \begin{array}{l} 1\le i_1<\cdots<i_p\le n-1\\ i_j-k_j\ge 1 \quad \forall\, j=1,\ldots,p \end{array} \right\}
$$

In particular, $\mathcal{H}_n(q)$ is a free R-module of rank $n!$.

2.2. **The Temperley–Lieb algebra $\mathrm{TL}_n(q)$.** Let $i=1,\ldots,n-2$. We set

$$
G_{i,i+1} := 1+qG_i+qG_{i+1}+q^2G_iG_{i+1}+q^2G_{i+1}G_i+q^3G_iG_{i+1}G_i = \sum_{w\in\langle s_i,s_{i+1}\rangle} q^{\ell(w)}G_w .
$$

We define the *Temperley–Lieb algebra* $\mathrm{TL}_n(q)$ to be the quotient $\mathcal{H}_n(q)/I_n$, where I_n is the ideal generated by the element $G_{1,2}$ (if $n\le 2$, we take $I_n=\{0\}$). We have $G_{i,i+1}\in I_n$ for all $i=1,\ldots,n-2$, since

$$
G_{i,i+1} = (G_1G_2\ldots G_{n-1})^{i-1}\, G_{1,2}\, (G_1G_2\ldots G_{n-1})^{-(i-1)}.
$$

Jones [Jo1] has shown that the set

$$
\mathcal{B}_{\mathrm{TL}_n(q)} := \left\{ (G_{i_1}G_{i_1-1}\ldots G_{i_1-k_1})\ldots(G_{i_p}G_{i_p-1}\ldots G_{i_p-k_p}) \;\middle|\; \begin{array}{l} 1\le i_1<\cdots<i_p\le n-1\\ 1\le i_1-k_1<\cdots<i_p-k_p\le n-1 \end{array} \right\}
$$

is a basis of $\mathrm{TL}_n(q)$ as an R-module. In particular, $\mathrm{TL}_n(q)$ is a free R-module of rank C_n, where C_n denotes the *n-th Catalan number*, that is,

$$
C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}^2 .
$$

2.3. **The Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q)$.** The *Yokonuma–Hecke algebra of type A*, denoted by $\mathrm{Y}_{d,n}(q)$, is an R-associative algebra generated by the elements

$$
g_1,\ldots,g_{n-1},t_1,\ldots,t_n
$$

subject to the following relations:

$$
\begin{array}{lrcll}
(\mathrm{b}_1) & g_ig_j & = & g_jg_i & \text{for all } i,j=1,\ldots,n-1 \text{ with } |i-j|>1,\\
(\mathrm{b}_2) & g_ig_{i+1}g_i & = & g_{i+1}g_ig_{i+1} & \text{for all } i=1,\ldots,n-2,\\
(\mathrm{f}_1) & t_it_j & = & t_jt_i & \text{for all } i,j=1,\ldots,n,\\
(\mathrm{f}_2) & t_jg_i & = & g_it_{s_i(j)} & \text{for all } i=1,\ldots,n-1 \text{ and } j=1,\ldots,n,\\
(\mathrm{f}_3) & t_j^d & = & 1 & \text{for all } j=1,\ldots,n,
\end{array} \tag{2.3}
$$

where s_i denotes the transposition $(i,i+1)$, together with the quadratic relations:

$$
g_i^2 = 1+(q-q^{-1})\,e_i\,g_i \qquad \text{for all } i=1,\ldots,n-1, \tag{2.4}
$$

where

$$e_i := \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}. \tag{2.5}$$

Note that we have $e_i^2 = e_i$ and $e_i g_i = g_i e_i$ for all $i = 1, \ldots, n-1$. Moreover, we have

$$t_i e_i = t_{i+1} e_i \qquad \text{for all } i = 1, \ldots, n-1. \tag{2.6}$$

Remark 2.3. If we specialise q to 1, the defining relations (2.3)–(2.4) become the defining relations for the complex reflection group $G(d,1,n) \cong (\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$. Thus, the algebra $\mathrm{Y}_{d,n}(q)$ is a deformation of $\mathbb{C}[G(d,1,n)]$. Moreover, for $d = 1$, the Yokonuma–Hecke algebra $\mathrm{Y}_{1,n}(q)$ coincides with the Iwahori–Hecke algebra $\mathcal{H}_n(q)$ of type A.

Remark 2.4. The relations (b_1), (b_2), (f_1) and (f_2) are defining relations for the classical framed braid group $\mathcal{F}_n \cong \mathbb{Z} \wr B_n$, where B_n is the classical braid group on n strands, with the t_j's being interpreted as the "elementary framings" (framing 1 on the jth strand). The relations $t_j^d = 1$ mean that the framing of each braid strand is regarded modulo d. Thus, the algebra $\mathrm{Y}_{d,n}(q)$ arises naturally as a quotient of the framed braid group algebra $R[\mathcal{F}_n]$ over the modular relations (f_3) and the quadratic relations (2.4). Moreover, relations (2.3) are defining relations for the modular framed braid group $\mathcal{F}_{d,n} \cong (\mathbb{Z}/d\mathbb{Z}) \wr B_n$, so the algebra $\mathrm{Y}_{d,n}(q)$ can be also seen as a quotient of the modular framed braid group algebra $R[\mathcal{F}_{d,n}]$ over the quadratic relations (2.4).

Remark 2.5. The generators g_i satisfying the quadratic relation (2.4) were introduced in [ChPdA]. In all the papers [Ju2, JuLa2, JuLa3, JuLa4, ChLa, GJKL1, GJKL2] prior to [ChPdA], the authors consider the braid generators $\overline{g}_i := g_i + (q-1)\, e_i g_i$ (and thus, $g_i = \overline{g}_i + (q^{-1}-1)\, e_i \overline{g}_i$), which satisfy the quadratic relation

$$\overline{g}_i^2 = 1 + (q^2-1)\, e_i + (q^2-1)\, e_i\, \overline{g}_i\ , \tag{2.7}$$

and the Yokonuma–Hecke algebra is defined over the ring $\mathbb{C}[q^2, q^{-2}]$. Note that

$$e_i \overline{g}_i = q e_i g_i \qquad \text{for all } i = 1, \ldots, n-1. \tag{2.8}$$

Remark 2.6. In [ChPo2, ChPo3], we consider the braid generators $\widetilde{g}_i := q g_i$, which satisfy the quadratic relation

$$\widetilde{g}_i^2 = q^2 + (q^2-1)\, e_i\, \widetilde{g}_i\ , \tag{2.9}$$

and the Yokonuma–Hecke algebra is defined over the ring $\mathbb{C}[q^2, q^{-2}]$. Note that

$$e_i \widetilde{g}_i = q e_i g_i \qquad \text{for all } i = 1, \ldots, n-1. \tag{2.10}$$

Let $w \in \mathfrak{S}_n$ and let $w = s_{i_1} s_{i_2} \ldots s_{i_r}$ be a reduced expression for w. By Matsumoto's lemma, the element $g_w := g_{i_1} g_{i_2} \ldots g_{i_r}$ is well defined. Juyumaya [Ju2] has shown that the set

$$\mathcal{B}_{\mathrm{Y}_{d,n}(q)} := \{ t_1^{a_1} t_2^{a_2} \ldots t_n^{a_n} g_w \,|\, 0 \leq a_1, a_2, \ldots, a_n \leq d-1,\ w \in \mathfrak{S}_n \}$$

forms a basis of $\mathrm{Y}_{d,n}(q)$ over R, which is called the *standard basis*. In particular, $\mathrm{Y}_{d,n}(q)$ is a free R-module of rank $d^n n!$.

2.4. **The Framisation of the Temperley–Lieb algebra $\mathrm{FTL}_{d,n}(q)$.** Let $i = 1, \ldots, n-2$. We set

$$g_{i,i+1} := 1 + q g_i + q g_{i+1} + q^2 g_i g_{i+1} + q^2 g_{i+1} g_i + q^3 g_i g_{i+1} g_i = \sum_{w \in \langle s_i, s_{i+1} \rangle} q^{\ell(w)} g_w.$$

We define the *Framisation of the Temperley–Lieb algebra* to be the quotient $\mathrm{Y}_{d,n}(q)/I_{d,n}$, where $I_{d,n}$ is the ideal generated by the element $e_1 e_2\, g_{1,2}$ (if $n \leq 2$, we take $I_{d,n} = \{0\}$). Note that, due to (2.6), the product $e_1 e_2$ commutes with g_1 and with g_2, so it commutes with $g_{1,2}$. Further, we have $e_i e_{i+1} g_{i,i+1} \in I_{d,n}$ for all $i = 1, \ldots, n-2$, since

$$e_i e_{i+1} g_{i,i+1} = (g_1 g_2 \ldots g_{n-1})^{i-1}\, e_1 e_2\, g_{1,2}\, (g_1 g_2 \ldots g_{n-1})^{-(i-1)}.$$

Remark 2.7. The ideal $I_{d,n}$ is also generated by the element $\sum_{0 \leq a,b \leq d-1} t_1^a t_2^b t_3^{-a-b}\, g_{1,2}$.

Remark 2.8. For $d = 1$, the Framisation of the Temperley–Lieb algebra $\mathrm{FTL}_{1,n}(q)$ coincides with the classical Temperley–Lieb algebra $\mathrm{TL}_n(q)$.

Remark 2.9. In [GJKL2], the Framisation of the Temperley–Lieb algebra is defined to be the quotient $\mathrm{Y}_{d,n}(q)/\overline{I}_{d,n}$, where $\overline{I}_{d,n}$ is the ideal generated by the element $e_1e_2\,\overline{g}_{1,2}$, where

$$\overline{g}_{1,2} = 1 + \overline{g}_1 + \overline{g}_2 + \overline{g}_1\overline{g}_2 + \overline{g}_2\overline{g}_1 + \overline{g}_1\overline{g}_2\overline{g}_1.$$

Due to (2.8) and the fact that the e_i's are idempotents, we have $e_1e_2\,\overline{g}_{1,2} = e_1e_2\,g_{1,2}$, and so $I_{d,n} = \overline{I}_{d,n}$.

Remark 2.10. In [ChPo2, ChPo3], we define the Framisation of the Temperley–Lieb algebra to be the quotient $\mathrm{Y}_{d,n}(q)/\widetilde{I}_{d,n}$, where $\widetilde{I}_{d,n}$ is the ideal generated by the element $e_1e_2\,\widetilde{g}_{1,2}$, where

$$\widetilde{g}_{1,2} = 1 + \widetilde{g}_1 + \widetilde{g}_2 + \widetilde{g}_1\widetilde{g}_2 + \widetilde{g}_2\widetilde{g}_1 + \widetilde{g}_1\widetilde{g}_2\widetilde{g}_1.$$

Due to (2.10) and the fact that the e_i's are idempotents, we have $e_1e_2\,\widetilde{g}_{1,2} = e_1e_2\,g_{1,2}$, and so $I_{d,n} = \widetilde{I}_{d,n}$.

3. An isomorphism theorem for the Framisation of the Temperley–Lieb algebra

Lusztig has proved that Yokonuma–Hecke algebras are isomorphic to direct sums of matrix algebras over certain subalgebras of classical Iwahori–Hecke algebras [Lu, §34]. For the Yokonuma–Hecke algebras $\mathrm{Y}_{d,n}(q)$, these are all tensor products of Iwahori–Hecke algebras of type A. This result was reproved in [JaPdA] using the presentation of $\mathrm{Y}_{d,n}(q)$ given by Juyumaya. Since we use the same presentation, we will use the latter exposition of the result in order to prove an analogous statement for $\mathrm{FTL}_{d,n}(q)$.

3.1. **Compositions and Young subgroups.** Let $\mu \in \mathrm{Comp}_d(n)$, where

$$\mathrm{Comp}_d(n) = \{\mu = (\mu_1, \mu_2, \ldots, \mu_d) \in \mathbb{N}^d \mid \mu_1 + \mu_2 + \cdots + \mu_d = n\}.$$

We say that μ is a *composition of n with d parts*. The Young subgroup $\mathfrak{S}_\mu$ of $\mathfrak{S}_n$ is the subgroup $\mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_d}$, where $\mathfrak{S}_{\mu_1}$ acts on the letters $\{1, \ldots, \mu_1\}$, $\mathfrak{S}_{\mu_2}$ acts on the letters $\{\mu_1 + 1, \ldots, \mu_1 + \mu_2\}$, and so on. Thus, $\mathfrak{S}_\mu$ is a parabolic subgroup of $\mathfrak{S}_n$ generated by the transpositions $s_j = (j, j+1)$ with $j \in J^\mu := \{1, \ldots, n-1\} \setminus \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_{d-1}\}$.

We have an Iwahori–Hecke algebra $\mathcal{H}^\mu(q)$ associated with $\mathfrak{S}_\mu$, which is the subalgebra of $\mathcal{H}_n(q)$ generated by $\{G_j \mid j \in J^\mu\}$. The algebra $\mathcal{H}^\mu(q)$ is a free R-module with basis $\{G_w \mid w \in \mathfrak{S}_\mu\}$, and it is isomorphic to the tensor product (over R) of Iwahori–Hecke algebras $\mathcal{H}_{\mu_1}(q) \otimes \mathcal{H}_{\mu_2}(q) \otimes \cdots \otimes \mathcal{H}_{\mu_d}(q)$ (with $\mathcal{H}_{\mu_i}(q) \cong R$ if $\mu_i \leq 1$).

For $i = 1, \ldots, d$, we denote by ρ_{μ_i} the natural surjection $\mathcal{H}_{\mu_i}(q) \twoheadrightarrow \mathcal{H}_{\mu_i}(q)/I_{\mu_i} \cong \mathrm{TL}_{\mu_i}(q)$, where I_{μ_i} is the ideal generated by $G_{\mu_1+\cdots+\mu_{i-1}+1,\mu_1+\cdots+\mu_{i-1}+2}$ if $\mu_i > 2$ and $I_{\mu_i} = \{0\}$ if $\mu_i \leq 2$. We obtain that $\rho^\mu := \rho_{\mu_1} \otimes \rho_{\mu_2} \otimes \cdots \otimes \rho_{\mu_d}$ is a surjective R-algebra homomorphism $\mathcal{H}^\mu(q) \twoheadrightarrow \mathrm{TL}^\mu(q)$, where $\mathrm{TL}^\mu(q)$ denotes the tensor product of Temperley–Lieb algebras $\mathrm{TL}_{\mu_1}(q) \otimes \mathrm{TL}_{\mu_2}(q) \otimes \cdots \otimes \mathrm{TL}_{\mu_d}(q)$.

3.2. **An isomorphism theorem for the Yokonuma–Hecke algebra** $\mathrm{Y}_{d,n}(q)$**.** Let $\{\xi_1, \ldots, \xi_d\}$ be the set of all d-th roots of unity (ordered arbitrarily). Let χ be an irreducible character of the abelian group $\mathcal{A}_{d,n} \cong (\mathbb{Z}/d\mathbb{Z})^n$ generated by the elements $t_1, t_2, \ldots, t_n$. There exists a primitive idempotent of $\mathbb{C}[\mathcal{A}_{d,n}]$ associated with χ defined as

$$E_\chi := \prod_{j=1}^{n} \left(\frac{1}{d} \sum_{s=0}^{d-1} \chi(t_j^s) t_j^{d-s} \right) = \prod_{j=1}^{n} \left(\frac{1}{d} \sum_{s=0}^{d-1} \chi(t_j)^s t_j^{d-s} \right).$$

Moreover, we can define a composition $\mu^\chi \in \mathrm{Comp}_d(n)$ by setting

$$\mu_i^\chi := \#\{j \in \{1, \ldots, n\} \mid \chi(t_j) = \xi_i\} \quad \text{for all } i = 1, \ldots, d.$$

Conversely, given a composition $\mu \in \mathrm{Comp}_d(n)$, we can consider the subset $\mathrm{Irr}^\mu(\mathcal{A}_{d,n})$ of $\mathrm{Irr}(\mathcal{A}_{d,n})$ defined as

$$\mathrm{Irr}^\mu(\mathcal{A}_{d,n}) := \{\chi \in \mathrm{Irr}(\mathcal{A}_{d,n}) \mid \mu^\chi = \mu\}.$$

There is an action of $\mathfrak{S}_n$ on $\mathrm{Irr}^\mu(\mathcal{A}_{d,n})$ given by

$$w(\chi)(t_j) := \chi(t_{w^{-1}(j)}) \quad \text{for all } w \in \mathfrak{S}_n,\ j = 1, \ldots, n.$$

Let $\chi_1^\mu \in \mathrm{Irr}^\mu(\mathcal{A}_{d,n})$ be the character given by

$$\begin{cases} \chi_1^\mu(t_1) & = & \cdots & = & \chi_1^\mu(t_{\mu_1}) & = & \xi_1 \\ \chi_1^\mu(t_{\mu_1+1}) & = & \cdots & = & \chi_1^\mu(t_{\mu_1+\mu_2}) & = & \xi_2 \\ \chi_1^\mu(t_{\mu_1+\mu_2+1}) & = & \cdots & = & \chi_1^\mu(t_{\mu_1+\mu_2+\mu_3}) & = & \xi_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_1^\mu(t_{\mu_1+\cdots+\mu_{d-1}+1}) & = & \cdots & = & \chi_1^\mu(t_n) & = & \xi_d \end{cases}$$

The stabiliser of χ_1^μ under the action of $\mathfrak{S}_n$ is the Young subgroup $\mathfrak{S}_\mu$. In each left coset in $\mathfrak{S}_n/\mathfrak{S}_\mu$, we can take a representative of minimal length; such a representative is unique (see, for example, [GePf, §2.1]). Let

$$\{\pi_{\mu,1}, \pi_{\mu,2}, \ldots, \pi_{\mu,m_\mu}\}$$

be this set of distinguished left coset representatives of $\mathfrak{S}_n/\mathfrak{S}_\mu$, with

$$m_\mu = \frac{n!}{\mu_1!\mu_2!\ldots\mu_d!}$$

and the convention that $\pi_{\mu,1} = 1$. Then, if we set

$$\chi_k^\mu := \pi_{\mu,k}(\chi_1^\mu) \quad \text{for all } k = 1, \ldots, m_\mu,$$

we have

$$\mathrm{Irr}^\mu(\mathcal{A}_{d,n}) = \{\chi_1^\mu, \chi_2^\mu, \ldots, \chi_{m_\mu}^\mu\}.$$

We now set

$$E_\mu := \sum_{\chi \in \mathrm{Irr}^\mu(\mathcal{A}_{d,n})} E_\chi = \sum_{k=1}^{m_\mu} E_{\chi_k^\mu}.$$

Since the set $\{E_\chi \,|\, \chi \in \mathrm{Irr}(\mathcal{A}_{d,n})\}$ forms a complete set of orthogonal idempotents in $\mathrm{Y}_{d,n}(q)$, and

$$t_j E_\chi = E_\chi t_j = \chi(t_j) E_\chi \quad \text{and} \quad g_w E_\chi = E_{w(\chi)} g_w \tag{3.1}$$

for all $\chi \in \mathrm{Irr}(\mathcal{A}_{d,n})$, $j = 1, \ldots, n$ and $w \in \mathfrak{S}_n$, we have that the set $\{E_\mu \,|\, \mu \in \mathrm{Comp}_d(n)\}$ forms a complete set of central orthogonal idempotents in $\mathrm{Y}_{d,n}(q)$ (cf. [JaPdA, §2.4]). In particular, we have the following decomposition of $\mathrm{Y}_{d,n}(q)$ into a direct sum of two-sided ideals:

$$\mathrm{Y}_{d,n}(q) = \bigoplus_{\mu \in \mathrm{Comp}_d(n)} E_\mu \mathrm{Y}_{d,n}(q).$$

We can now define an R-linear map

$$\Psi_\mu : E_\mu \mathrm{Y}_{d,n}(q) \to \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q))$$

as follows: for all $k \in \{1, \ldots, m_\mu\}$ and $w \in \mathfrak{S}_n$, we set

$$\Psi_\mu(E_{\chi_k^\mu} g_w) := G_{\pi_{\mu,k}^{-1} w \pi_{\mu,l}} M_{k,l}\,,$$

where $l \in \{1, \ldots, m_\mu\}$ is uniquely defined by the relation $w(\chi_l^\mu) = \chi_k^\mu$ and $M_{k,l}$ is the elementary $m_\mu \times m_\mu$ matrix with 1 in position (k,l). Note that $\pi_{\mu,k}^{-1} w \pi_{\mu,l} \in \mathfrak{S}_\mu$. We also define an R-linear map

$$\Phi_\mu : \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q)) \to E_\mu \mathrm{Y}_{d,n}(q)$$

as follows: for all $k, l \in \{1, \ldots, m_\mu\}$ and $w \in \mathfrak{S}_\mu$, we set

$$\Phi_\mu(G_w M_{k,l}) := E_{\chi_k^\mu} g_{\pi_{\mu,k} w \pi_{\mu,l}^{-1}} E_{\chi_l^\mu}.$$

Then we have the following [JaPdA, Theorem 3.1]:

Theorem 3.1. *Let $\mu \in \mathrm{Comp}_d(n)$. The linear map Ψ_μ is an isomorphism of R-algebras with inverse map Φ_μ. As a consequence, the map*

$$\Psi_{d,n} := \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \Psi_\mu : \mathrm{Y}_{d,n}(q) \to \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q))$$

is also an isomorphism of R-algebras, with inverse map

$$\Phi_{d,n} := \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \Phi_\mu : \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q)) \to \mathrm{Y}_{d,n}(q).$$

Remark 3.2. In [ChPo3], we show that we can construct similar isomorphisms over the smaller ring $\mathbb{C}[q^2, q^{-2}]$ when we consider the generators $\widetilde{g}_i := qg_i$ and $\widetilde{G}_i := qG_i$. Note that

$$\Psi_\mu(E_{\chi_k^\mu}\widetilde{g}_w) := q^{\ell(w)-\ell(\pi_{\mu,k}^{-1}w\pi_{\mu,l})}\widetilde{G}_{\pi_{\mu,k}^{-1}w\pi_{\mu,l}}M_{k,l}$$

and

$$\Phi_\mu(\widetilde{G}_w M_{k,l}) := q^{\ell(w)-\ell(\pi_{\mu,k}^{-1}w\pi_{\mu,l})}E_{\chi_k^\mu}\widetilde{g}_{\pi_{\mu,k}w\pi_{\mu,l}^{-1}}E_{\chi_l^\mu}.$$

In order to do this, we make use of Deodhar's lemma (see, for example, [GePf, Lemma 2.1.2]) about the distinguished left coset representatives of $\mathfrak{S}_n/\mathfrak{S}_\mu$:

Lemma 3.3. (Deodhar's lemma) *Let $\mu \in \mathrm{Comp}_d(n)$. For all $k \in \{1, \ldots, m_\mu\}$ and $i = 1, \ldots, n-1$, let $l \in \{1, \ldots, m_\mu\}$ be uniquely defined by the relation $s_i(\chi_l^\mu) = \chi_k^\mu$. We have*

$$\pi_{\mu,k}^{-1}s_i\pi_{\mu,l} = \begin{cases} 1 & \text{if } k \neq l; \\ \\ s_j & \text{if } k = l, \end{cases}$$

for some $j \in J^\mu$.

Deodhar's lemma implies that, for all $i = 1, \ldots, n-1$, $\Psi_\mu(E_\mu\widetilde{g}_i)$ is a symmetric matrix whose diagonal non-zero coefficients are of the form $\widetilde{G}_j$ with $j \in J^\mu$, while all non-diagonal non-zero coefficients are equal to q. Thus, if consider the diagonal matrix

$$U_\mu := \sum_{k=1}^{m_\mu} q^{\ell(\pi_{\mu,k})}M_{k,k},$$

the coefficients of the matrix $U_\mu\Psi_\mu(E_\mu\widetilde{g}_i)U_\mu^{-1}$ satisfy:

$$(U_\mu\Psi_\mu(E_\mu\widetilde{g}_i)U_\mu^{-1})_{k,l} = q^{(\ell(\pi_{\mu,k})-\ell(\pi_{\mu,l}))}(\Psi_\mu(E_\mu\widetilde{g}_i))_{k,l}\,,$$

for all $k, l \in \{1, \ldots, m_\mu\}$. Therefore, following the definition of Ψ_μ and Deodhar's lemma, the matrix $U_\mu\Psi_\mu(E_\mu\widetilde{g}_i)U_\mu^{-1}$ is a matrix whose diagonal coefficients are the same as the diagonal coefficients of $\Psi_\mu(E_\mu\widetilde{g}_i)$ (and thus of the form $\widetilde{G}_j$ with $j \in J^\mu$), while all non-diagonal non-zero coefficients are equal to either 1 or q^2. Moreover, since, for all $j = 1, \ldots, n$,

$$\Psi_\mu(E_\mu t_j) = \sum_{k=1}^{m_\mu} \chi_k^\mu(t_j)M_{k,k}$$

is a diagonal matrix, we have $U_\mu\Psi_\mu(E_\mu t_j)U_\mu^{-1} = \Psi_\mu(E_\mu t_j)$. We conclude that the map

$$\widetilde{\Psi}_\mu : E_\mu\mathrm{Y}_{d,n}(q) \to \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q))$$

defined by

$$\widetilde{\Psi}_\mu(E_\mu a) := U_\mu\Psi_\mu(E_\mu a)U_\mu^{-1},$$

for all $a \in \mathrm{Y}_{d,n}(q)$, is an isomorphism of $\mathbb{C}[q^2, q^{-2}]$-algebras. Its inverse is the map

$$\widetilde{\Phi}_\mu : \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q)) \to E_\mu\mathrm{Y}_{d,n}(q)$$

defined by

$$\widetilde{\Phi}_\mu(A) := \Phi_\mu(U_\mu^{-1}AU_\mu),$$

for all $A \in \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q))$. As a consequence, the map

$$\widetilde{\Psi}_{d,n} := \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \widetilde{\Psi}_\mu : \mathrm{Y}_{d,n}(q) \to \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q))$$

is also an isomorphism of $\mathbb{C}[q^2, q^{-2}]$-algebras, with inverse map

$$\widetilde{\Phi}_{d,n} := \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \widetilde{\Phi}_\mu : \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q)) \to \mathrm{Y}_{d,n}(q).$$

3.3. **From $\mathrm{FTL}_{d,n}(q)$ to Temperley–Lieb.** Recall that $\mathrm{FTL}_{d,n}(q)$ is the quotient $\mathrm{Y}_{d,n}(q)/I_{d,n}$, where $I_{d,n}$ is the ideal generated by the element $e_1e_2\, g_{1,2}$ (with $I_{d,n} = \{0\}$ if $n \leq 2$). Let $\mu \in \mathrm{Comp}_d(n)$. We will study the image of $e_1e_2\, g_{1,2}$ under the isomorphism Ψ_μ.

By (3.1), for all $i = 1, \ldots, n-1$ and $\chi \in \mathrm{Irr}(\mathcal{A}_{d,n})$, we have

$$e_iE_\chi = E_\chi e_i = \frac{1}{d}\sum_{s=0}^{d-1} \chi(t_i)^s \chi(t_{i+1})^{d-s} E_\chi = \begin{cases} E_\chi & \text{if } \chi(t_i) = \chi(t_{i+1}); \\ \\ 0 & \text{if } \chi(t_i) \neq \chi(t_{i+1}). \end{cases} \tag{3.2}$$

We deduce that, for all $k = 1, \ldots, m_\mu$,

$$E_{\chi_k^\mu} e_1e_2g_{1,2} = \begin{cases} E_{\chi_k^\mu} g_{1,2} & \text{if } \chi_k^\mu(t_1) = \chi_k^\mu(t_2) = \chi_k^\mu(t_3); \\ \\ 0 & \text{otherwise .} \end{cases} \tag{3.3}$$

Proposition 3.4. *Let $\mu \in \mathrm{Comp}_d(n)$ and $k \in \{1, \ldots, m_\mu\}$. We have*

$$\Psi_\mu(E_{\chi_k^\mu} e_1e_2g_{1,2}) = \begin{cases} G_{i,i+1}M_{k,k} & \textit{for some } i \in \{1, \ldots, n-2\} \;\; \textit{if } \chi_k^\mu(t_1) = \chi_k^\mu(t_2) = \chi_k^\mu(t_3); \\ \\ 0 & \textit{otherwise .} \end{cases}$$

Thus, $\Psi_\mu(E_\mu e_1e_2g_{1,2})$ is a diagonal matrix in $\mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q))$ with all non-zero coefficients being of the form $G_{i,i+1}$ for some $i \in \{1, \ldots, n-2\}$.

Proof. If $\chi_k^\mu(t_1) = \chi_k^\mu(t_2) = \chi_k^\mu(t_3)$, then $w(\chi_k^\mu) = \chi_k^\mu$ for all $w \in \langle s_1, s_2\rangle \subseteq \mathfrak{S}_n$, and so

$$\Psi_\mu(E_{\chi_k^\mu} g_{1,2}) = \sum_{w \in \langle s_1,s_2\rangle} \Psi_\mu(E_{\chi_k^\mu} g_w) = \sum_{w \in \langle s_1,s_2\rangle} G_{\pi_{\mu,k}^{-1} w \pi_{\mu,k}} M_{k,k}. \tag{3.4}$$

We will show that there exists $i \in \{1, \ldots, n-2\}$ such that

$$\sum_{w \in \langle s_1,s_2\rangle} G_{\pi_{\mu,k}^{-1} w \pi_{\mu,k}} = G_{i,i+1}.$$

By Lemma 3.3, there exist $i, j \in J^\mu$ such that

$$\pi_{\mu,k}^{-1} s_1 \pi_{\mu,k} = s_i \quad \text{and} \quad \pi_{\mu,k}^{-1} s_2 \pi_{\mu,k} = s_j.$$

Consequently, $\pi_{\mu,k}^{-1} s_1s_2 \pi_{\mu,k} = s_is_j$, $\pi_{\mu,k}^{-1} s_2s_1 \pi_{\mu,k} = s_js_i$ and $\pi_{\mu,k}^{-1} s_1s_2s_1 \pi_{\mu,k} = s_is_js_i$. Moreover, since s_1 and s_2 do not commute, s_i and s_j do not commute either, so we must have $j \in \{i-1, i+1\}$. Hence, if $j = i-1$, then

$$\sum_{w \in \langle s_1,s_2\rangle} G_{\pi_{\mu,k}^{-1} w \pi_{\mu,k}} = G_{i-1,i},$$

while if $j = i+1$, then

$$\sum_{w \in \langle s_1,s_2\rangle} G_{\pi_{\mu,k}^{-1} w \pi_{\mu,k}} = G_{i,i+1}.$$

We conclude that there exists $i \in \{1, \ldots, n-2\}$ such that

$$\sum_{w \in \langle s_1,s_2\rangle} G_{\pi_{\mu,k}^{-1} w \pi_{\mu,k}} = G_{i,i+1},$$

whence we deduce that

$$\Psi_\mu(E_{\chi_k^\mu} g_{1,2}) = G_{i,i+1}M_{k,k}.$$

Combining this with (3.3) yields the desired result. ∎

Example 3.5. Let us consider the case $d = 2$ and $n = 4$. We have

$$(\mu, m_\mu) \in \{((4,0),1), ((3,1),4), ((2,2),6), ((1,3),4), ((0,4),1)\}.$$

Then

$$\Psi_\mu(E_{\chi_k^\mu} e_1 e_2 g_{1,2}) = \begin{cases} G_{1,2} & \text{if } \mu = (4,0) \text{ or } \mu = (0,4) \ , \\ G_{1,2}\, M_{1,1} & \text{if } \mu = (3,1) \text{ and } k = 1 \ , \\ G_{2,3}\, M_{4,4} & \text{if } \mu = (1,3) \text{ and } k = 4 \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

where we take $\pi_{(1,3),4} = s_3 s_2 s_1$.

Now, recall the surjective R-algebra homomorphism $\rho^\mu : \mathcal{H}^\mu(q) \twoheadrightarrow \mathrm{TL}^\mu(q)$ defined in §3.1. The map ρ^μ induces a surjective R-algebra homomorphism $\mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q)) \twoheadrightarrow \mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q))$, which we also denote by ρ^μ. We obtain that

$$\rho^\mu \circ \Psi_\mu : E_\mu \mathrm{Y}_{d,n}(q) \to \mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q))$$

is a surjective R-algebra homomorphism.

In order for $\rho^\mu \circ \Psi_\mu$ to factor through $E_\mu \mathrm{Y}_{d,n}(q)/E_\mu I_{d,n} \cong E_\mu \mathrm{FTL}_{d,n}(q)$, all elements of $E_\mu I_{d,n}$ have to belong to the kernel of $\rho^\mu \circ \Psi_\mu$. Since $I_{d,n}$ is the ideal generated by the element $e_1 e_2 g_{1,2}$, it is enough to show that $(\rho^\mu \circ \Psi_\mu)(e_1 e_2 g_{1,2}) = 0$. This is immediate by Proposition 3.4. Hence, if we denote by ϱ^μ the natural surjection $E_\mu \mathrm{Y}_{d,n}(q) \twoheadrightarrow E_\mu \mathrm{Y}_{d,n}(q)/E_\mu I_{d,n} \cong E_\mu \mathrm{FTL}_{d,n}(q)$, there exists a unique R-algebra homomorphism $\psi_\mu : E_\mu \mathrm{FTL}_{d,n}(q) \to \mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q))$ such that the following diagram is commutative:

$$\begin{array}{ccc} E_\mu \mathrm{Y}_{d,n}(q) & \xrightarrow{\Psi_\mu} & \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q)) \\ \downarrow{\scriptstyle \varrho^\mu} & & \downarrow{\scriptstyle \rho^\mu} \\ E_\mu \mathrm{FTL}_{d,n}(q) & \xrightarrow{\psi_\mu} & \mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q)) \end{array} \tag{3.5}$$

Since $\rho^\mu \circ \Psi_\mu$ is surjective, ψ_μ is also surjective.

3.4. **From Temperley–Lieb to** $\mathrm{FTL}_{d,n}(q)$**.** We now consider the surjective R-algebra homomorphism:

$$\varrho^\mu \circ \Phi_\mu : \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q)) \to E_\mu \mathrm{FTL}_{d,n}(q),$$

where Φ_μ is the inverse of Ψ_μ. In order for $\varrho^\mu \circ \Phi_\mu$ to factor through $\mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q))$, we have to show that $G_{i,i+1} M_{k,l}$ belongs to the kernel of $\varrho^\mu \circ \Phi_\mu$ for all $i = 1, \ldots, n-2$ such that $G_{i,i+1} \in \mathcal{H}^\mu(q)$ (that is, $\{i, i+1\} \subseteq J^\mu$) and for all $k, l \in \{1, \ldots, m_\mu\}$. Since

$$G_{i,i+1} M_{k,l} = M_{k,1} G_{i,i+1} M_{1,1} M_{1,l}$$

and $\varrho^\mu \circ \Phi_\mu$ is an homomorphism of R-algebras, it is enough to show that $(\varrho^\mu \circ \Phi_\mu)(G_{i,i+1} M_{1,1}) = 0$.

Let $i = 1, \ldots, n-2$ such that $G_{i,i+1} \in \mathcal{H}^\mu(q)$. By definition of Φ_μ, and since $\pi_{\mu,1} = 1$, we have

$$\Phi_\mu(G_{i,i+1} M_{1,1}) = E_{\chi_1^\mu} g_{i,i+1} E_{\chi_1^\mu}. \tag{3.6}$$

Now, since $G_{i,i+1} \in \mathcal{H}^\mu(q)$, there exists $j \in \{1, \ldots, d\}$ such that $\mu_j > 2$ and $G_{i,i+1} \in \mathcal{H}_{\mu_j}(q)$, that is, $i \in \{\mu_1 + \cdots + \mu_{j-1} + 1, \ldots, \mu_1 + \cdots + \mu_{j-1} + \mu_j - 2\}$. By definition of χ_1^μ, we have

$$\chi_1^\mu(t_{\mu_1+\cdots+\mu_{j-1}+1}) = \cdots = \chi_1^\mu(t_{\mu_1+\cdots+\mu_{j-1}+\mu_j}) = \xi_j,$$

whence

$$\chi_1^\mu(t_i) = \chi_1^\mu(t_{i+1}) = \chi_1^\mu(t_{i+2}) = \xi_j.$$

Following (3.2), we obtain

$$\Phi_\mu(G_{i,i+1} M_{1,1}) = E_{\chi_1^\mu} g_{i,i+1} E_{\chi_1^\mu} = E_{\chi_1^\mu} e_i e_{i+1} g_{i,i+1} E_{\chi_1^\mu}.$$

Since $e_i e_{i+1} g_{i,i+1} \in I_{d,n}$, we deduce that $(\varrho^\mu \circ \Phi_\mu)(G_{i,i+1} M_{1,1}) = 0$, as desired.

We conclude that there exists a unique R-algebra homomorphism $\phi_\mu : \mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q)) \to E_\mu \mathrm{FTL}_{d,n}(q)$ such that the following diagram is commutative:

$$\begin{tikzcd}
E_\mu \mathrm{Y}_{d,n}(q) \arrow[d, "\varrho^\mu"] & \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q)) \arrow[l, "\Phi_\mu"'] \arrow[d, "\rho^\mu"] \\
E_\mu \mathrm{FTL}_{d,n}(q) & \mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q)) \arrow[l, "\phi_\mu"']
\end{tikzcd} \tag{3.7}$$

Since $\varrho^\mu \circ \Phi_\mu$ is surjective, ϕ_μ is also surjective.

3.5. **An isomorphism theorem for the Framisation of the Temperley–Lieb algebra** $\mathrm{FTL}_{d,n}(q)$**.** We are now ready to prove the main result of this section.

Theorem 3.6. *Let $\mu \in \mathrm{Comp}_d(n)$. The linear map ψ_μ is an isomorphism of R-algebras with inverse map ϕ_μ. As a consequence, the map*

$$\psi_{d,n} := \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \psi_\mu : \mathrm{FTL}_{d,n}(q) \to \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q))$$

is also an isomorphism of R-algebras, with inverse map

$$\phi_{d,n} := \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \phi_\mu : \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q)) \to \mathrm{FTL}_{d,n}(q).$$

Proof. Since the diagrams (3.5) and (3.7) are commutative, we have

$$\rho^\mu \circ \Psi_\mu = \psi_\mu \circ \varrho^\mu \quad \text{and} \quad \varrho^\mu \circ \Phi_\mu = \phi_\mu \circ \rho^\mu.$$

This implies that

$$\rho^\mu \circ \Psi_\mu \circ \Phi_\mu = \psi_\mu \circ \phi_\mu \circ \rho^\mu \quad \text{and} \quad \varrho^\mu \circ \Phi_\mu \circ \Psi_\mu = \phi_\mu \circ \psi_\mu \circ \varrho^\mu.$$

By Theorem 3.1, $\Psi_\mu \circ \Phi_\mu = \mathrm{id}_{\mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu(q))}$ and $\Phi_\mu \circ \Psi_\mu = \mathrm{id}_{E_\mu \mathrm{Y}_{d,n}(q)}$, whence

$$\rho^\mu = \psi_\mu \circ \phi_\mu \circ \rho^\mu \quad \text{and} \quad \varrho^\mu = \phi_\mu \circ \psi_\mu \circ \varrho^\mu.$$

Since the maps ρ^μ and ϱ^μ are surjective, we obtain

$$\psi_\mu \circ \phi_\mu = \mathrm{id}_{\mathrm{Mat}_{m_\mu}(\mathrm{TL}^\mu(q))} \quad \text{and} \quad \phi_\mu \circ \psi_\mu = \mathrm{id}_{E_\mu \mathrm{FTL}_{d,n}(q)},$$

as desired. □

Remark 3.7. In [ChPo3], we show that we can construct similar isomorphisms over the smaller ring $\mathbb{C}[q^2, q^{-2}]$ when we consider the generators $\widetilde{g}_i = qg_i$ and $\widetilde{G}_i = qG_i$. For this, we use the presentation of $\mathrm{FTL}_{d,n}(q)$ given in Remark 2.10 and the isomorphisms $\widetilde{\Psi}_\mu$ and $\widetilde{\Phi}_\mu$ defined in Remark 3.2.

3.6. **A basis for the Framisation of the Temperley–Lieb algebra** $\mathrm{FTL}_{d,n}(q)$**.** We recall that in §2.2 we defined a basis $\mathcal{B}_{\mathrm{TL}_n(q)}$ for the Temperley–Lieb algebra $\mathrm{TL}_n(q)$. Thanks to Theorem 3.6, we obtain the following basis for $\mathrm{FTL}_{d,n}(q)$ as an R-module:

Proposition 3.8. *The set*

$$\left\{ \phi_\mu(b_1 b_2 \dots b_d\, M_{k,l}) \,|\, \mu \in \mathrm{Comp}_d(n), b_i \in \mathcal{B}_{\mathrm{TL}_{\mu_i}(q)} \text{ for all } i = 1, \dots, d, 1 \leq k, l \leq m_\mu \right\}$$

is a basis of $\mathrm{FTL}_{d,n}(q)$ as an R-module. In particular, $\mathrm{FTL}_{d,n}(q)$ is a free R-module of rank

$$\sum_{\mu \in \mathrm{Comp}_d(n)} m_\mu^2\, C_{\mu_1} C_{\mu_2} \cdots C_{\mu_d}.$$

4. Markov traces and link invariants

The presentation for the Temperley–Lieb algebra given in §2.2 is due to Jones, who used a Markov trace defined on it, the *Jones–Ocneanu trace*, to construct his famous polynomial invariant for classical links, the *Jones polynomial* [Jo2]. A similar construction on the Framisation of the Temperley–Lieb algebra yields invariants for framed and classical links [GJKL2]. In this section, we will relate the latter to the Jones polynomial using the isomorphism of Theorem 3.6.

4.1. **The inductive Jones–Ocneanu trace.** Using the natural algebra inclusions $\mathcal{H}_n(q) \subset \mathcal{H}_{n+1}(q)$ for $n \in \mathbb{N}$ (setting $\mathcal{H}_n(q) := R$ for $n \leq 1$), we can define the Jones–Ocneanu trace on $\bigcup_{n\geq 0} \mathcal{H}_n(q)$ as follows [Jo2, Theorem 5.1]:

Theorem 4.1. *Let z be an indeterminate over $\mathbb{C}$. There exists a unique linear Markov trace*

$$\tau_z : \bigcup_{n\geq 0} \mathcal{H}_n(q) \longrightarrow R[z]$$

defined inductively on $\mathcal{H}_n(q)$, for all $n \geq 0$, by the following rules:

$$\begin{array}{rclll} \tau_z(1) & = & 1 & & 1 \in \mathcal{H}_n(q) \\ \tau_z(ab) & = & \tau_z(ba) & & a, b \in \mathcal{H}_n(q) \\ \tau_z(aG_n) & = & z\,\tau_z(a) & & a \in \mathcal{H}_n(q). \end{array}$$

It is easy to check (by solving the equation $\tau_z(G_{1,2}) = 0$) that the trace τ_z passes to the quotient Temperley–Lieb algebra $\mathrm{TL}_n(q)$ if and only if

$$z = -\frac{1}{q^2(q+q^{-1})} = -\frac{1}{q^3+1} \quad \text{or} \quad z = -q^{-1} .$$

The second value is discarded as not being topologically interesting. For $z = -(q^3+1)^{-1}$, we will simply denote τ_z by τ.

Recall that we denote by ρ_n the natural surjection $\mathcal{H}_n(q) \twoheadrightarrow \mathcal{H}_n(q)/I_n \cong \mathrm{TL}_n(q)$. Let us denote by $\sigma_1, \ldots, \sigma_{n-1}$ the generators of the classical braid group B_n, such that the natural epimorphism $\delta_n : RB_n \twoheadrightarrow \mathcal{H}_n(q)$ is given by $\delta_n(\sigma_i) = G_i$. Then $\rho_n \circ \delta_n : RB_n \twoheadrightarrow \mathrm{TL}_n(q)$ is also an epimorphism.

Let now $\mathcal{L}$ denote the set of oriented links. For any $\alpha \in B_n$, we denote by $\widehat{\alpha}$ the link obtained as the closure of α. By the Alexander Theorem, we have $\mathcal{L} = \cup_n \{\widehat{\alpha} \,|\, \alpha \in B_n\}$. Further, by the Markov Theorem, isotopy of links is generated by conjugation in B_n ($\alpha\beta \sim \beta\alpha$) and by positive and negative stabilisation ($\alpha \sim \alpha\sigma_n^{\pm 1}$). Jones's method for constructing polynomial link invariants consists of normalising and re-scaling τ with respect to the latter: For any $\alpha \in B_n$, let

$$V_q(\widehat{\alpha}) := (-q-q^{-1})^{n-1} q^{2\epsilon(\alpha)} \, (\tau \circ \rho_n \circ \delta_n)(\alpha) \ ,$$

where $\epsilon(\alpha)$ is the sum of the exponents of the braiding generators σ_i in the word α. Then the map

$$V_q : \mathcal{L} \to R, \ \widehat{\alpha} \mapsto V_q(\widehat{\alpha})$$

is an 1-variable ambient isotopy invariant of oriented links, known as the *Jones polynomial* (cf. [Jo2]).

Example 4.2. We consider the Hopf link with two positive crossings, which is the closure of the braid $\sigma_1^2 \in B_2$. We have

$$V_q(\widehat{\sigma_1^2}) = (-q-q^{-1})q^4\tau(G_1^2) = -(q+q^{-1})q^4\left(1 - \frac{q-q^{-1}}{q^2(q+q^{-1})}\right) = -(q+q^{-1})q^4 + (q-q^{-1})q^2 = -q^5 - q.$$

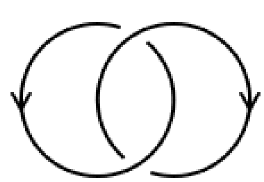

Figure 1. The Hopf link with two positive crossings.

Remark 4.3. More generally, for any value of z, the trace τ_z can be normalised and re-scaled with respect to positive and negative stabilisation as follows: For any $\alpha \in B_n$, let

$$P_{q,z}(\widehat{\alpha}) := \Lambda_H^{n-1}(\sqrt{\lambda_H})^{\epsilon(\alpha)}\,(\tau_z \circ \delta_n)(\alpha)\ ,$$

where

$$\lambda_H := \frac{z-(q-q^{-1})}{z} \quad \text{and} \quad \Lambda_H := \frac{1}{z\sqrt{\lambda_H}}.$$

Then the map

$$P_{q,z} : \mathcal{L} \to R[z^{\pm 1}, \sqrt{\lambda_H}^{\pm 1}],\ \widehat{\alpha} \mapsto P_{q,z}(\widehat{\alpha})$$

is a 2-variable invariant of oriented links, known as the *HOMFLYPT polynomial* (cf. [HOMFLY, PT]). For $z = -(q^3+1)^{-1}$, we get $\lambda_H = q^4$ and $\Lambda_H = -q-q^{-1}$, whence $P_{q,z} = V_q$.

4.2. **The stabilised Jones–Ocneanu traces.** Instead of normalising and re-scaling the Jones–Ocneanu trace τ, we can consider a family of traces $\tau^n : \mathcal{H}_n(q) \to R$ for $n \in \mathbb{N}$ that are stabilised by definition. However, for any $a \in \mathcal{H}_n(q)$, we have $\tau^n(a) \neq \tau^{n+1}(a)$.

More specifically, let us consider the Iwahori–Hecke algebra $\mathcal{H}_n(q)$ with braid generators $G_i' := q^2 G_i$. These satisfy the quadratic relation

$$G_i'^{\,2} = q^4 + q^2(q-q^{-1})G_i'. \tag{4.1}$$

We then have the following (see, for example, [GePf, Theorem 4.5.2]):

Theorem 4.4. *There exists a unique family of R-linear Markov traces $\tau^n : \mathcal{H}_n(q) \to R$ such that*

$$\begin{array}{rclcll} \tau^1(1) & = & 1 & & & \\ \tau^n(ab) & = & \tau^n(ba) & & & a,b \in \mathcal{H}_n(q) \\ \tau^{n+1}(aG_n') & = & \tau^{n+1}(aG_n'^{\,-1}) & = & \tau^n(a) & a \in \mathcal{H}_n(q). \end{array}$$

Moreover, we have $\tau^{n+1}(a) = (-q-q^{-1})\tau^n(a)$ for all $a \in \mathcal{H}_n(q)$.

We observe that

$$G_{1,2} = 1 + q^{-1}G_1' + q^{-1}G_2' + q^{-2}G_1'G_2' + q^{-2}G_2'G_1' + q^{-3}G_1'G_2'G_1'.$$

We have

$$\begin{array}{rcl} \tau^3(1) & = & (-q-q^{-1})^2\tau^1(1) = q^2+2+q^{-2} \\ \tau^3(G_1') & = & (-q-q^{-1})\tau^2(G_1') = (-q-q^{-1})\tau^1(1) = -q-q^{-1} \\ \tau^3(G_2') & = & \tau^2(1) = (-q-q^{-1})\tau^1(1) = -q-q^{-1} \\ \tau^3(G_1'G_2') & = & \tau^2(G_1') = \tau^1(1) = 1 \\ \tau^3(G_2'G_1') & = & \tau^2(G_1') = \tau^1(1) = 1 \\ \tau^3(G_1'G_2'G_1') & = & \tau^2(G_1'^{\,2}) = q^4\tau^2(1) + q^2(q-q^{-1})\tau^2(G_1') = -q^5-q \end{array}$$

whence

$$\tau^3(G_{1,2}) = q^2+2+q^{-2}-2-2q^{-2}+2q^{-2}-q^2-q^{-2} = 0.$$

Since we have

$$\tau^n(G_{1,2}) = (-q-q^{-1})^{n-3}\tau^3(G_{1,2}),$$

the trace τ^n factors through the Temperley–Lieb algebra $\mathrm{TL}_n(q)$ for all $n \in \mathbb{N}$. Further, if we consider the natural epimorphism $\delta_n' : RB_n \twoheadrightarrow \mathcal{H}_n(q)$ given by $\delta'(\sigma_i) = G_i'$, we have [Jo2, §11]:

$$(\tau^n \circ \rho_n \circ \delta_n')(\alpha) = V_q(\widehat{\alpha}) \quad \text{for all} \ \ \alpha \in B_n. \tag{4.2}$$

Example 4.5. We have

$$(\tau^2 \circ \rho_2 \circ \delta_2')(\sigma_1^2) = \tau^2(G_1'^{\,2}) = -q^5-q.$$

Remark 4.6. More generally, for any value of z, if we consider the braid generators $G_i' := \sqrt{\lambda_H}G_i$, where $\lambda_H = \frac{z-(q-q^{-1})}{z}$, and we define a family of stabilised Jones–Ocneanu traces $(\tau_z^n)_{n\in\mathbb{N}}$ as in Theorem 4.4, with $\tau_z^{n+1}(a) = (\sqrt{z\lambda_H})^{-1}\tau_z^n(a)$ and with values in $R[z^{\pm 1}, \sqrt{\lambda_H}^{\pm 1}]$, then we have [Jo2, (6.2)]:

$$(\tau_z^n \circ \delta_n')(\alpha) = P_{q,z}(\widehat{\alpha}) \quad \text{for all} \ \ \alpha \in B_n.$$

4.3. **The inductive Juyumaya trace.** An important property of the Yokonuma–Hecke algebra is that it also supports a Markov trace defined for all values of n. More precisely, due to the inclusions $\mathrm{Y}_{d,n}(q) \subset \mathrm{Y}_{d,n+1}(q)$ (setting $\mathrm{Y}_{d,0}(q) := R$), we obtain (cf. [Ju1, Theorem 12]):

Theorem 4.7. *Let z, $x_1, \ldots, x_{d-1}$ be indeterminates over $\mathbb{C}$. There exists a unique linear Markov trace*

$$\mathrm{tr}_{d,z} : \bigcup_{n\geq 0} \mathrm{Y}_{d,n}(q) \longrightarrow \mathbb{C}[z, x_1, \ldots, x_{d-1}]$$

defined inductively on $\mathrm{Y}_{d,n}(q)$, for all $n \geq 0$, by the following rules:

$$\begin{array}{rclll}
\mathrm{tr}_{d,z}(1) & = & 1 & 1 \in \mathrm{Y}_{d,n}(q) & \\
\mathrm{tr}_{d,z}(ab) & = & \mathrm{tr}_{d,z}(ba) & a, b \in \mathrm{Y}_{d,n}(q) & \\
\mathrm{tr}_{d,z}(ag_n) & = & z\, \mathrm{tr}_{d,z}(a) & a \in \mathrm{Y}_{d,n}(q) & \\
\mathrm{tr}_{d,z}(at_{n+1}^k) & = & x_k\, \mathrm{tr}_{d,z}(a) & a \in \mathrm{Y}_{d,n}(q) & (1 \leq k \leq d-1).
\end{array}$$

Remark 4.8. Note that, for $d = 1$, the trace $\mathrm{tr}_{1,z}$ is defined by only the first three rules. Thus, tr_1 coincides with the Jones–Ocneanu trace τ_z on the Iwahori–Hecke algebra $\mathcal{H}_n(q) \cong \mathrm{Y}_{1,n}(q)$.

The values of the parameters for which the trace $\mathrm{tr}_{d,z}$ passes to the quotient algebra $\mathrm{FTL}_{d,n}(q)$ are given in [GJKL2, Theorem 6]; their determination is not straightforward as in the classical case. However, not all of them are topologically interesting.

First, let us denote by $\varrho_{d,n}$ the natural surjection $\mathrm{Y}_{d,n}(q) \twoheadrightarrow \mathrm{Y}_{d,n}(q)/I_{d,n} \cong \mathrm{FTL}_{d,n}(q)$. Recall that we denote by $\mathcal{F}_n$ the classical framed braid group. We have $\mathcal{F}_n \cong \mathbb{Z} \wr B_n$, and there exists a natural epimorphism $\gamma_{d,n} : R\mathcal{F}_n \twoheadrightarrow \mathrm{Y}_{d,n}(q)$ given by $\gamma_{d,n}(\sigma_i) = g_i$ and $\gamma_{d,n}(t_j^k) = t_j^{k (\mathrm{mod}\, d)}$ for all $k \in \mathbb{Z}$. The map $\varrho_{d,n} \circ \gamma_{d,n} : R\mathcal{F}_n \twoheadrightarrow \mathrm{FTL}_{d,n}(q)$ is also an algebra epimorphism.

Let now $\mathcal{L}_f$ denote the set of oriented framed links. By the Alexander Theorem, we have $\mathcal{L}_f = \cup_n \{\widehat{\alpha} \,|\, \alpha \in \mathcal{F}_n\}$. Further, by the Markov Theorem for framed links [KoSm, Lemma 1], isotopy of framed links is generated by conjugation in $\mathcal{F}_n$ ($\alpha\beta \sim \beta\alpha$) and by positive and negative stabilisation ($\alpha \sim \alpha\sigma_n^{\pm 1}$), for any n. In view of all this, Juyumaya and Lambropoulou [JuLa2] attempted to normalise and re-scale the trace $\mathrm{tr}_{d,z}$ in order to obtain invariants for framed knots and links following Jones's method; they discovered that this is the only Markov trace known in literature that cannot be re-scaled directly. They showed that $\mathrm{tr}_{d,z}$ re-scales when the parameters $(x_k)_{1\leq k\leq d-1}$ satisfy the following system of equations, known as the *E-system*:

$$\sum_{s=0}^{d-1} x_{k+s} x_{d-s} = x_k \sum_{s=0}^{d-1} x_s x_{d-s} \qquad (1 \leq k \leq d-1), \tag{4.3}$$

with $x_0 = x_d = 1$. The solutions of the E-system where computed by Gérardin in the Appendix of [GJKL2] and they are parametrised by the non-empty subsets of $\mathbb{Z}/d\mathbb{Z}$: If D is such a subset, then

$$x_k = \frac{1}{|D|} \sum_{j \in D} \exp\left(\frac{2\pi i j k}{d}\right) \qquad (1 \leq k \leq d-1).$$

For the rest of the paper, D will denote a non-empty subset of $\mathbb{Z}/d\mathbb{Z}$ and $(x_1, \ldots, x_{d-1})$ will be the corresponding solution of the E-system. We will denote by $\mathrm{tr}_{d,D,z}$ the Juyumaya trace with these parameters and we will call it the *specialised Juyumaya trace*. We have $\mathrm{tr}_{d,D,z}(e_i) = 1/|D| =: E_D$ for all i. According to [GJKL2, (7.7)], the trace $\mathrm{tr}_{d,D,z}$ passes to the quotient algebra $\mathrm{FTL}_{d,n}(q)$ if and only if

$$z = -\frac{E_D}{q^2(q+q^{-1})} = -\frac{E_D}{q^3+1} \quad \text{or} \quad z = -\frac{E_D}{q}.$$

The second value is discarded as not being topologically interesting. For $z = -E_D(q^3+1)^{-1}$, we will simply denote $\mathrm{tr}_{d,D,z}$ by $\mathrm{tr}_{d,D}$. Normalising and re-scaling $\mathrm{tr}_{d,D}$ with respect to positive and negative stabilisation yields the following: For any $\alpha \in \mathcal{F}_n$, let

$$\phi_{d,D,q}(\widehat{\alpha}) := \left(-\frac{q+q^{-1}}{E_D}\right)^{n-1} q^{2\epsilon(\alpha)} \left(\mathrm{tr}_{d,D} \circ \varrho_{d,n} \circ \gamma_{d,n}\right)(\alpha)\ ,$$

where $\epsilon(\alpha)$ is the sum of the exponents of the braiding generators σ_i in the word α. Then the map

$$\phi_{d,D,q} : \mathcal{L}_f \to R,\ \widehat{\alpha} \mapsto \phi_{d,D,q}(\widehat{\alpha})$$

is an 1-variable ambient isotopy invariant of oriented framed links [GJKL2, (7.8)].

We denote by $\theta_{d,D,q}$ the restriction of $\phi_{d,D,q}$ to the set $\mathcal{L}$ of classical links; the map $\theta_{d,D,q}$ is an 1-variable ambient isotopy invariant of oriented classical links.

Example 4.9. We consider the classical Hopf link with two positive crossings. We have

$$\theta_{d,D,q}(\widehat{\sigma_1^2}) = \left(-\frac{q+q^{-1}}{E_D}\right) q^4 \mathrm{tr}_{d,D}(g_1^2) = \left(-\frac{q+q^{-1}}{E_D}\right) q^4 \left(1 - \frac{(q-q^{-1})E_D}{q^2(q+q^{-1})}\right) = -\frac{q^5+q^3}{E_D} + q^3 - q.$$

We now consider the framed Hopf link with framings 0 and 1. This is the closure of the framed braid $t_2\sigma_1^2$. Note that $(\mathrm{tr}_{d,D} \circ \varrho_{d,n} \circ \gamma_{d,n})(t_2\sigma_1^2) = \mathrm{tr}_{d,D}(t_2 g_1^2) = \mathrm{tr}_{d,D}(g_1 t_1 g_1) = \mathrm{tr}_{d,D}(t_1 g_1^2) = (\mathrm{tr}_{d,D} \circ \varrho_{d,n} \circ \gamma_{d,n})(t_1\sigma_1^2)$. We have

$$\mathrm{tr}_{d,D}(t_1 g_1^2) = \mathrm{tr}_{d,D}(t_1) + (q-q^{-1})\mathrm{tr}_{d,D}(t_1 e_1 g_1) = \mathrm{tr}_{d,D}(t_1)\left(1 - \frac{(q-q^{-1})E_D}{q^2(q+q^{-1})}\right) = x_1 \mathrm{tr}_{d,D}(g_1^2).$$

We deduce that

$$\phi_{d,D,q}(\widehat{t_2\sigma_1^2}) = \left(-\frac{q+q^{-1}}{E_D}\right) q^4 \mathrm{tr}_{d,D}(t_2 g_1^2) = x_1 \theta_{d,D,q}(\sigma_1^2) = x_1\left(-\frac{q^5+q^3}{E_D} + q^3 - q\right).$$

Remark 4.10. More generally, for any value of z, the trace $\mathrm{tr}_{d,D,z}$ can be normalised and re-scaled with respect to positive and negative stabilisation as follows: For any $\alpha \in \mathcal{F}_n$, let

$$\Phi_{d,D,q,z}(\widehat{\alpha}) := \Lambda_D^{n-1}(\sqrt{\lambda_D})^{\epsilon(\alpha)}\,(\mathrm{tr}_{d,D,z} \circ \gamma_{d,n})(\alpha)\ ,$$

where

$$\lambda_D := \frac{z-(q-q^{-1})E_D}{z} \quad \text{and} \quad \Lambda_D := \frac{1}{z\sqrt{\lambda_D}}.$$

Then the map

$$\Phi_{d,D,q,z} : \mathcal{L}_f \to R[z^{\pm 1}, \sqrt{\lambda_D}^{\pm 1}],\ \widehat{\alpha} \mapsto \Phi_{d,D,q,z}(\widehat{\alpha})$$

is a 2-variable invariant of oriented framed links [CJKL, Theorem 3.1]. For $z = -E_D(q^3+1)^{-1}$, we get $\lambda_D = q^4$ and $\Lambda_D = -(q+q^{-1})/E_D$, whence $\Phi_{d,D,q,z} = \phi_{d,D,q}$.

We denote by $\Theta_{d,D,q,z}$ the restriction of $\Phi_{d,D,q,z}$ to the set $\mathcal{L}$ of classical links; the map $\Theta_{d,D,q,z}$ is a 2-variable invariant of oriented classical links. For $z = -E_D(q^3+1)^{-1}$, we have $\Theta_{d,D,q,z} = \theta_{d,D,q}$.

Remark 4.11. Using the same construction, but replacing the generators g_i with the generators $\overline{g}_i := g_i + (q-1)\, e_i g_i$, Juyumaya and Lambropoulou defined 2-variable invariants for framed [JuLa2] and classical [JuLa3] links from the specialised Juyumaya trace on the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q)$. Considering the specialised Juyumaya trace on $\mathrm{FTL}_{d,n}(q)$, but replacing again g_i with $\overline{g}_i$, Goundaroulis, Juyumaya, Kontogeorgis and Lambropoulou defined 1-variable invariants for framed and classical links in [GJKL2]. As shown in [CJKL, Section 8], these invariants are not topologically equivalent to the ones we define in this paper. There is no such issue when replacing g_i with $\widetilde{g}_i := qg_i$ or with $g_i' := q^2 g_i$.

Remark 4.12. For $d = 1$, we have $\theta_{1,\{0\},q} = V_q$ and $\Theta_{1,\{0\},q,z} = P_{q,z}$. More generally, when $|D| = 1$, it was shown in [ChLa] that the invariants $\theta_{d,D,q}$ and $\Theta_{d,D,q,z}$ are equivalent to the Jones and HOMFLYPT polynomials respectively.

4.4. **The stabilised Jacon–Poulain d'Andecy traces.** Similarly to the Jones–Ocneanu trace, instead of normalising and re-scaling $\mathrm{tr}_{d,D}$, we can consider a family of traces $\mathrm{tr}^n_{d,D} : \mathrm{Y}_{d,n}(q) \to R$ for $n \in \mathbb{N}$ that are stabilised by definition. However, for any $a \in \mathrm{Y}_{d,n}(q)$, we have $\mathrm{tr}^n_{d,D}(a) \neq \mathrm{tr}^{n+1}_{d,D}(a)$.

More specifically, let us consider the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q)$ with braid generators $g_i' := q^2 g_i$. These satisfy the quadratic relation

$$g_i'^2 = q^4 + q^2(q-q^{-1})e_i g_i'. \tag{4.4}$$

We then have the following (see also [JaPdA, §5.2], [PdA, §5.2]):

Theorem 4.13. *There exists a unique family of R-linear Markov traces $\mathrm{tr}_{d,D}^n : \mathrm{Y}_{d,n}(q) \to R$ such that*

$$\begin{array}{rcllll} \mathrm{tr}_{d,D}^1(1) & = & 1 & & & \\ \mathrm{tr}_{d,D}^n(ab) & = & \mathrm{tr}_{d,D}^n(ba) & & a, b \in \mathrm{Y}_{d,n}(q) & \\ \mathrm{tr}_{d,D}^{n+1}(ag'_n) & = & \mathrm{tr}_{d,D}^{n+1}(a{g'_n}^{-1}) & = \quad \mathrm{tr}_{d,D}^n(a) & a \in \mathrm{Y}_{d,n}(q) & \\ \mathrm{tr}_{d,D}^{n+1}(at_{n+1}^k) & = & x_k \mathrm{tr}_{d,D}^{n+1}(a) & & a \in \mathrm{Y}_{d,n}(q) & (1 \leq k \leq d-1). \end{array}$$

Moreover, we have $\mathrm{tr}_{d,D}^{n+1}(a) = (-q - q^{-1})E_D^{-1}\mathrm{tr}_{d,D}^n(a)$ for all $a \in \mathrm{Y}_{d,n}(q)$.

First of all, note that

$$q^4 {g'_n}^{-1} = g'_n - q^2(q - q^{-1})e_n.$$

Therefore, for all $a \in \mathrm{Y}_{d,n}(q)$, we have

$$q^2(q - q^{-1})\mathrm{tr}_{d,D}^{n+1}(ae_n) = \mathrm{tr}_{d,D}^{n+1}(ag'_n) - q^4\mathrm{tr}_{d,D}^{n+1}(a{g'_n}^{-1}) = (1 - q^4)\mathrm{tr}_{d,D}^n(a),$$

whence

$$\mathrm{tr}_{d,D}^{n+1}(ae_n) = (-q - q^{-1})\mathrm{tr}_{d,D}^n(a) = E_D\mathrm{tr}_{d,D}^{n+1}(a). \tag{4.5}$$

Moreover,

$$\mathrm{tr}_{d,D}^{n+1}(ae_ng'_n) = \frac{1}{d}\sum_{s=0}^{d-1}\mathrm{tr}_{d,D}^{n+1}(at_n^s g'_n t_n^{d-s}) = \frac{1}{d}\sum_{s=0}^{d-1}\mathrm{tr}_{d,D}^n(at_n^s t_n^{d-s}) = \mathrm{tr}_{d,D}^n(a). \tag{4.6}$$

Now, we observe that

$$g_{1,2} = 1 + q^{-1}g'_1 + q^{-1}g'_2 + q^{-2}g'_1g'_2 + q^{-2}g'_2g'_1 + q^{-3}g'_1g'_2g'_1.$$

We have

$$\begin{array}{rcl} \mathrm{tr}_{d,D}^3(e_1e_2) & = & (-q - q^{-1})\mathrm{tr}_{d,D}^2(e_1) = (-q - q^{-1})^2\mathrm{tr}_{d,D}^1(1) = q^2 + 2 + q^{-2} \\ \mathrm{tr}_{d,D}^3(e_1e_2g'_1) & = & (-q - q^{-1})\mathrm{tr}_{d,D}^2(e_1g'_1) = (-q - q^{-1})\mathrm{tr}_{d,D}^1(1) = -q - q^{-1} \\ \mathrm{tr}_{d,D}^3(e_1e_2g'_2) & = & \mathrm{tr}_{d,D}^2(e_1) = (-q - q^{-1})\mathrm{tr}_{d,D}^1(1) = -q - q^{-1} \\ \mathrm{tr}_{d,D}^3(e_1e_2g'_1g'_2) & = & \mathrm{tr}_{d,D}^2(e_1g'_1) = \mathrm{tr}_{d,D}^1(1) = 1 \\ \mathrm{tr}_{d,D}^3(e_1e_2g'_2g'_1) & = & \mathrm{tr}_{d,D}^2(e_1g'_1) = \mathrm{tr}_{d,D}^1(1) = 1 \\ \mathrm{tr}_{d,D}^3(e_1e_2g'_1g'_2g'_1) & = & \mathrm{tr}_{d,D}^2(e_1{g'_1}^2) = q^4\mathrm{tr}_{d,D}^2(e_1) + q^2(q - q^{-1})\mathrm{tr}_{d,D}^2(e_1g'_1) = -q^5 - q \end{array}$$

whence

$$\mathrm{tr}_{d,D}^3(e_1e_2g_{1,2}) = q^2 + 2 + q^{-2} - 2 - 2q^{-2} + 2q^{-2} - q^2 - q^{-2} = 0.$$

Since we have

$$\mathrm{tr}_{d,D}^n(e_1e_2g_{1,2}) = \left(-\frac{q + q^{-1}}{E_D}\right)^{n-3}\mathrm{tr}_{d,D}^3(e_1e_2g_{1,2}),$$

the trace $\mathrm{tr}_{d,D}^n$ factors through the Framisation of the Temperley–Lieb algebra $\mathrm{FTL}_{d,n}(q)$ for all $n \in \mathbb{N}$. Further, if we consider the natural epimorphism $\gamma'_{d,n} : R\mathcal{F}_n \twoheadrightarrow \mathrm{Y}_{d,n}(q)$ given by $\gamma'_{d,n}(\sigma_i) = g'_i$ and $\gamma'_{d,n}(t_j^k) = t_j^{k (\mathrm{mod}\, d)}$ for all $k \in \mathbb{Z}$, we have [PdA, Remarks 5.4]:

$$(\mathrm{tr}_{d,D}^n \circ \varrho_{d,n} \circ \gamma'_{d,n})(\alpha) = \phi_{d,D,q}(\widehat{\alpha}) \quad \text{for all} \quad \alpha \in \mathcal{F}_n. \tag{4.7}$$

Example 4.14. We have

$$(\mathrm{tr}_{d,D}^2 \circ \varrho_{d,2} \circ \gamma'_{d,2})(\sigma_1^2) = \mathrm{tr}_{d,D}^2({g'_1}^2) = \mathrm{tr}_{d,D}^2(q^4 + q^2(q - q^{-1})e_1g'_1) = -\frac{q^5 + q^3}{E_D} + q^3 - q.$$

and

$$(\mathrm{tr}_{d,D}^2 \circ \varrho_{d,2} \circ \gamma'_{d,2})(t_2\sigma_1^2) = \mathrm{tr}_{d,D}^2(t_1{g'_1}^2) = q^4\mathrm{tr}_{d,D}^2(t_1) + q^2(q - q^{-1})\mathrm{tr}_{d,D}^1(t_1) = x_1\left(-\frac{q^5 + q^3}{E_D} + q^3 - q\right).$$

Remark 4.15. More generally, for any value of z, if we consider the braid generators $g_i' := \sqrt{\lambda_D} g_i$, where $\lambda_D = \frac{z-(q-q^{-1})E_D}{z}$, and we define a family of stabilised Jones–Ocneanu traces $(\mathrm{tr}^n_{d,D,z})_{n\in\mathbb{N}}$ as in Theorem 4.13, with $\mathrm{tr}^{n+1}_{d,D,z}(a) = (\sqrt{z\lambda_D})^{-1}\mathrm{tr}^n_{d,D,z}(a)$ and with values in $R[z^{\pm1}, \sqrt{\lambda_D}^{\pm1}]$, then we have [PdA, Remarks 5.4]:

$$(\mathrm{tr}^n_{d,D,z} \circ \gamma'_{d,n})(\alpha) = \Phi_{d,D,q,z}(\widehat{\alpha}) \quad \text{for all} \quad \alpha \in \mathcal{F}_n.$$

4.5. **Connecting the invariants with the use of the isomorphism theorem.** In this last subsection, we will only be interested in invariants of classical links. The invariants $\Theta_{d,D,q,z}$ and $\theta_{d,D,q}$ of §4.3 have been further studied in [CJKL] and [GoLa] respectively. where their following properties have been proved:

- (P1) They do not depend on d and D, but only on the cardinality of D (and equivalently on E_D).
- (P2) They can be generalised to skein link invariants where E_D is taken to be an indeterminate.
- (P3) They are not topologically equivalent to the HOMFLYPT polynomial and the Jones polynomial respectively.

We will illustrate point (P3) for the invariant $\theta_{d,D,q}$ with the following example.

Example 4.16. We consider the link $L := LLL(0)$ of [EKT] with the orientation of Figure 2. This is a 3-component link, whose components are one left-handed trefoil (T) and 2 unknots (U1 and U2). The link L has the same Jones polynomial as the disjoint union of 3 unknots, even though it is not topologically equivalent to it. We have:

$$V_q(L) = (q+q^{-1})^2 = V_q(\widehat{1}_{B_3})$$

Now, the link L is the closure of the following braid:

$$\sigma_1^{-1}\sigma_2^2\sigma_3^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_2^{-3}\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2 \in B_5.$$

In order to compute $\theta_{d,D,q}$ on the closure of this braid, we used the program designed for this reason by Karvounis [Ka], which is available at `http://www.math.ntua.gr/~sofia/yokonuma`. We have that $\theta_{d,D,q}(L)$ is equal to:

$$V_q(L) + (E_D - 1)\frac{q+q^{-1}}{E_D^2 q^{11}}\left(E_D\left(q^{16} - 3q^{14} + 2q^{12} - 5q^{10} + 6q^8 - 4q^6 + 4q^4 - 5q^2 + 2\right) - q^{10} - q^8 - q^6 + q^2\right).$$

We observe that for $E_D = 1$, $\theta_{d,D,q}(L) = V_q(L)$. Moreover,

$$\theta_{d,D,q}(\widehat{1}_{B_3}) = \left(-\frac{q+q^{-1}}{E_D}\right)^2 = E_D^{-2}V_q(\widehat{1}_{B_3}),$$

and so $\theta_{d,D,q}$ distinguishes two links that the Jones polynomial cannot distinguish.

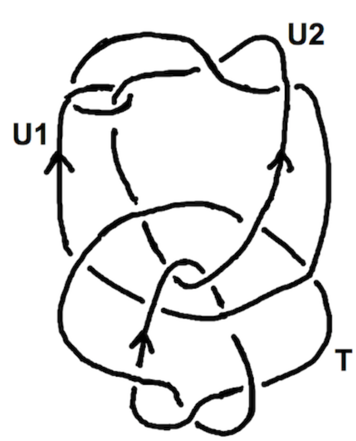

FIGURE 2. The link $LLL(0)$.

In the Appendix of [CJKL], Lickorish gave a closed combinatorial formula for computing the value of $\Theta_{d,D,q,z}$ on a link L which involves the HOMFLYPT polynomials of all sublinks of L and linking numbers [CJKL, Theorem B.1]. A specialisation of the above formula for $z = -E_D(q^3+1)^{-1}$ yields a similar result for the invariant $\theta_{d,D,q}$ [GoLa, Corollary 2]. Lickorish's formula for $\Theta_{d,D,q,z}$ was independently obtained by Poulain d'Andecy and Wagner [PdAWa] with the use of Theorem 3.1. In this section, we will obtain the corresponding formula for $\theta_{d,D,q}$ with the use of our Theorem 3.6.

First of all, due to property (P1), we can restrict our study to $\theta_{d,q} := \theta_{d,\mathbb{Z}/d\mathbb{Z},q}$. In this case, $E_D = 1/d$. We have already seen that the stabilised Jones–Ocneanu traces defined in Theorem 4.4 factor through the Temperley–Lieb algebra. Thus, one can define on

$$\bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathrm{TL}^{\mu}(q)) = \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathrm{TL}_{\mu_1}(q) \otimes \mathrm{TL}_{\mu_2}(q) \otimes \cdots \otimes \mathrm{TL}_{\mu_d}(q))$$

the trace

$$\sum_{\mu \in \mathrm{Comp}_d(n)} (\tau^{\mu_1} \otimes \tau^{\mu_2} \otimes \cdots \otimes \tau^{\mu_d}) \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}}$$

where $\mathrm{Tr}_{\mathrm{Mat}_{m_\mu}}$ denotes the usual trace of a matrix. By [JaPdA, §6], the map

$$T_{d,q} : \mathcal{L} \to R, \quad \widehat{\alpha} \mapsto \sum_{\mu \in \mathrm{Comp}_d(n)} (\tau^{\mu_1} \otimes \tau^{\mu_2} \otimes \cdots \otimes \tau^{\mu_d}) \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ (\psi_{d,n} \circ \varrho_{d,n} \circ \gamma'_{d,n})(\alpha)$$

is an 1-variable invariant of oriented classical links. This in turn implies that, for a given oriented link L, we have [PdAWa, Corollary 4.2]:

$$T_{d,q}(L) = d! \sum_{\pi} q^{4\nu(\pi)} V_q(\pi L) \tag{4.8}$$

where the sum is over all partitions π of the components of L into d (unordered) subsets, $V_q(\pi L)$ is the product of the Jones polynomials of the d sublinks of L defined by π and $\nu(\pi)$ is the sum of all linking numbers of pairs of components that are in distinct sets of π.

Remark 4.17. Note that the sum of linking numbers appearing in [PdAWa, Corollary 4.2] is twice the sum of linking numbers $\nu(\pi)$, as defined in [CJKL, Theorem B.1] and here.

We then obtain the following closed combinatorial formula for $\theta_{d,q}$.

Proposition 4.18. *Let L be an oriented link with m components. Then*

$$\theta_{d,q}(L) = \sum_{k=1}^{m} \frac{(d-1)(d-2)\cdots(d-k+1)}{k!} (-q-q^{-1})^{k-1} T_{k,q}(L) \tag{4.9}$$

Proof. Recall that $\theta_{d,q}(L) = (\mathrm{tr}^n_{d,\mathbb{Z}/d\mathbb{Z}} \circ \varrho_{d,n} \circ \gamma'_{d,n})(\alpha)$, where $\alpha \in B_n$ is such that $\widehat{\alpha} = L$. Then, by [PdA, Proposition 5.5], we have

$$\theta_{d,q}(L) = \frac{1}{d} \sum_{k=1}^{m} \binom{d}{k} (-q-q^{-1})^{k-1} T_{k,q}(L) = \sum_{k=1}^{m} \frac{(d-1)!}{k!(d-k)!} (-q-q^{-1})^{k-1} T_{k,q}(L),$$

and so (4.9) holds. □

Remark 4.19. Because of property (P2), Formula (4.9) is still valid if we replace the integer d by an indeterminate (corresponding to E_D^{-1}). The standard notation used for this generalised invariant is θ (cf. [GoLa]).

Example 4.20. We will use Formula (4.9) to compute the value of $\theta_{d,q}$ on the Hopf link with two positive crossings. The Hopf link has two components, each of them being an unknot, and linking number $ln(Hopf) = 1$. Formula (4.9) in combination with Equation (4.8) reads:

$$\begin{aligned} \theta_{d,q}(Hopf) &= V_q(Hopf) + (d-1)(-q-q^{-1}) q^{4ln(Hopf)} V_q(Unknot)^2 \\ &= -q^5 - q + d(-q^5 - q^3) + q^5 + q^3 = q^3 - q - d(q^5 + q^3) \end{aligned}$$

since $V_q(Unknot) = 1$. This coincides with the value that we found in Example 4.9 for $E_D = 1/d$.

Example 4.21. We will now use Formula (4.9) to compute the value of $\theta_{d,q}$ on $L := LLL(0)$ of Figure 2. We will denote by TU1 (respectively TU2) the 2-component link obtained when removing the component U2 (respectively U1) from L, and by $\mathrm{U}^{1,2}$ the 2-component link obtained when removing the component T from L. We have used the programming language SAGE [Sage] to compute the Jones polynomials of these three 2-component links, while it is easy to determine their linking numbers by hand. We have:

$$\begin{array}{rcll} V_q(\mathrm{TU1}) & = & -q^{-3}(q^{10}+q^6+q^2-1) & \text{and} \quad ln(\mathrm{TU1}) = 2 \\ V_q(\mathrm{TU2}) & = & -q^{-15}(q^{10}+q^6+q^2-1) & \text{and} \quad ln(\mathrm{TU2}) = -2 \\ V_q(\mathrm{U}^{1,2}) & = & q^{-3}(q^{10}+q^6+q^2-1)-2(q^5+q) & \text{and} \quad ln(\mathrm{U}^{1,2}) = 0. \end{array} \tag{4.10}$$

Formula (4.9) in combination with Equation (4.8) reads:

$$\begin{aligned} \theta_{d,q}(L) \;=\; & V_q(L)+ \\ & +(d-1)(-q-q^{-1})q^{4(ln(\mathrm{TU2})+ln(\mathrm{U}^{1,2}))}V_q(\mathrm{TU1})V_q(\mathrm{U2})+ \\ & +(d-1)(-q-q^{-1})q^{4(ln(\mathrm{TU1})+ln(\mathrm{U}^{1,2}))}V_q(\mathrm{TU2})V_q(\mathrm{U1})+ \\ & +(d-1)(-q-q^{-1})q^{4(ln(\mathrm{TU1})+ln(\mathrm{TU2}))}V_q(\mathrm{U}^{1,2})V_q(\mathrm{T})+ \\ & +(d-1)(d-2)(-q-q^{-1})^2q^{4(ln(\mathrm{TU1})+ln(\mathrm{TU2})+ln(\mathrm{U}^{1,2}))}V_q(\mathrm{T})V_q(\mathrm{U1})V_q(\mathrm{U2}) \end{aligned}$$

Using the fact that $V_q(\mathrm{U1}) = V_q(\mathrm{U2}) = 1$, since U1 and U2 are unknots, and replacing the linking numbers with their values from (4.10), we obtain that $\theta_{d,q}(L)$ is equal to:

$$V_q(L)-(d-1)(q+q^{-1})(q^{-8}V_q(\mathrm{TU1})+q^8V_q(\mathrm{TU2})+V_q(\mathrm{U}^{1,2})V_q(\mathrm{T}))+(d-1)(d-2)(q+q^{-1})^2V_q(\mathrm{T}).$$

Moreover, since T is a left-handed trefoil knot, we have $V_q(\mathrm{T}) = q^{-2}+q^{-6}-q^{-8}$. Using also the values for $V_q(\mathrm{TU1})$, $V_q(\mathrm{TU2})$ and $V_q(\mathrm{U}^{1,2})$ from (4.10), we calculate:

$$\begin{aligned} \theta_{d,q}(L) \;=\; & V_q(L)-(d-1)(q+q^{-1})(q^5-3q^3+2q-7q^{-1}+4q^{-3}-6q^{-5}+4q^{-7}-3q^{-9}+2q^{-11}) \\ & +(d-1)(d-2)(q+q^{-1})(q^{-1}+q^{-3}+q^{-5}-q^{-9}) \end{aligned}$$

which in turn is equal to:

$$V_q(L)-(d-1)(q+q^{-1})q^{-11}\left(q^{16}-3q^{14}+2q^{12}-5q^{10}+6q^8-4q^6+4q^4-5q^2+2-d(q^{10}+q^8+q^6-q^2)\right).$$

This coincides with the value that we found in Example 4.16 for $E_D = 1/d$.

Remark 4.22. It is obvious from the examples that, as the number of components becomes larger, the algebraic definition of $\theta_{d,q}$ directly from the Markov trace (or traces) on $\mathrm{FTL}_{d,n}(q)$ is more efficient computationally than its combinatorial definition with the use of Formula (4.9).

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