

REPRESENTATIONS OF THE FRAMISATION OF THE TEMPERLEY–LIEB ALGEBRA

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ABSTRACT. In this note, we describe the irreducible representations and give a dimension formula for the Framisation of the Temperley–Lieb algebra.

1. INTRODUCTION

The Temperley–Lieb algebra was introduced by Temperley and Lieb in [TeLi] for its applications in statistical mechanics. It was later shown by Jones [Jo1, Jo2] that it can be obtained as a quotient of the Iwahori–Hecke algebra of type A . Both algebras depend on a parameter q . Jones showed that there exists a unique Markov trace, called the Ocneanu trace, on the Iwahori–Hecke algebra, which depends on a parameter z . For a specific value of z , the Ocneanu trace passes to the Temperley–Lieb algebra. Jones used the Ocneanu trace on the Temperley–Lieb algebra to define a polynomial knot invariant, the famous Jones polynomial. Using the Ocneanu trace as defined originally on the Iwahori–Hecke algebra of type A yields another famous polynomial invariant, the HOMFLYPT polynomial, which is also known as the 2-variable Jones polynomial (the 2 variables being q and z).

Yokonuma–Hecke algebras were introduced by Yokonuma as generalisations of Iwahori–Hecke algebras in the context of finite Chevalley groups. The Yokonuma–Hecke algebra of type A is the centraliser algebra associated to the permutation representation of the general linear group over a finite field with respect to a maximal unipotent subgroup. Juyumaya has given a generic presentation for this algebra, depending on a parameter q , and defined a Markov trace on it, the latter depending on several parameters [Ju1, JuKa, Ju2]. This trace was subsequently used by Juyumaya and Lambropoulou for the construction of invariants for framed knots and links [JuLa1, JuLa2]. They later showed that these invariants can be also used for classical and singular knots and links [JuLa3, JuLa4]. The next step was to construct an analogue of the Temperley–Lieb algebra in this case.

As it is explained in more detail in [JuLa5], where the technique of framisation is thoroughly discussed, three possible candidates arose. The first candidate was the Yokonuma–Temperley–Lieb algebra, which was defined in [GJKL1] as the quotient of the Yokonuma–Hecke algebra by exactly the same ideal as the one used by Jones in the classical case. We studied the representation theory of this algebra and constructed a basis for it in [ChPo].

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The values of the parameters for which Juyumaya’s Markov trace passes to the Yokonuma–Temperley–Lieb algebra are given in [GJKL1]. Unfortunately, for these values, the invariants for classical knots and links obtained from the Yokonuma–Temperley–Lieb algebra are equivalent to the Jones polynomial.

A second candidate, which is more interesting topologically, was suggested in [GJKL2]. This is the Framisation of the Temperley–Lieb algebra, whose representation theory we study in this paper. The Framisation of the Temperley–Lieb algebra is defined in a subtler way than the Yokonuma–Temperley–Lieb algebra, as the quotient of the Yokonuma–Hecke algebra by a more elaborate ideal, and it is larger than the Yokonuma–Temperley–Lieb algebra. The values of the parameters for which Juyumaya’s Markov trace passes to this quotient are given in [GJKL2]. It was recently shown that the Juyumaya–Lambropoulou invariants for classical links are stronger than the HOMFLYPT polynomial [CJKL]. It turns out that, in a similar way, the invariants for classical links obtained from the Framisation of the Temperley–Lieb algebra are stronger than the Jones polynomial.

The third candidate is the so-called Complex Temperley–Lieb algebra, which is larger than the Framisation of the Temperley–Lieb algebra, but provides the same topological information (see [JuLa5]).

In this note, we study the representation theory of the Framisation of the Temperley–Lieb algebra. In Proposition 5 we give a complete description of its irreducible representations, by showing which irreducible representations of the Yokonuma–Hecke algebra pass to the quotient. The representations of the Yokonuma–Hecke algebra of type A were first studied by Thiem [Th1, Th2, Th3], but here we use their explicit description given later in [ChPA]. Our result generalises in a natural way the analogous result in the classical case. We then use the dimensions of the irreducible representations of the Framisation of the Temperley–Lieb algebra in order to compute the dimension of the algebra. We deduce a combinatorial formula involving Catalan numbers, given in Proposition 6.

A reference to the results of this note is included in [GJKL2], so we decided to finally provide them in written form. We also take this opportunity to write down the relations between three types of generators used in bibliography so far (Remark 2), and show that the Yokonuma–Hecke algebra is split semisimple over a smaller field than the one considered in [ChPA] (Corollary 2).

2. THE TEMPERLEY–LIEB ALGEBRA

In this section, we recall the definition of the Temperley–Lieb algebra as a quotient of the Iwahori–Hecke algebra of type A given by Jones [Jo2], and some classical results on its representation theory.

2.1. The Iwahori–Hecke algebra $\mathcal{H}_n(q)$. Let $n \in \mathbb{N}$ and let q be an indeterminate. The *Iwahori–Hecke algebra of type A* , denoted by $\mathcal{H}_n(q)$, is a $\mathbb{C}[q, q^{-1}]$ -associative algebra generated by the elements

$$G_1, \dots, G_{n-1}$$

subject to the following relations:

$$(2.1) \quad \begin{aligned} G_i G_j &= G_j G_i && \text{for all } i, j = 1, \dots, n-1 \text{ with } |i-j| > 1, \\ G_i G_{i+1} G_i &= G_{i+1} G_i G_{i+1} && \text{for all } i = 1, \dots, n-2, \end{aligned}$$

together with the quadratic relations:

$$(2.2) \quad G_i^2 = q + (q-1)G_i \quad \text{for all } i = 1, \dots, n-1.$$

Remark 1. If we specialise q to 1, the defining relations (2.1)–(2.2) become the defining relations for the symmetric group \mathfrak{S}_n . Thus the algebra $\mathcal{H}_n(q)$ is a deformation of the group algebra over \mathbb{C} of \mathfrak{S}_n .

2.2. The Temperley–Lieb algebra $\text{TL}_n(q)$. Let $i = 1, \dots, n-1$. We set

$$G_{i,i+1} := 1 + G_i + G_{i+1} + G_i G_{i+1} + G_{i+1} G_i + G_i G_{i+1} G_i.$$

We define the *Temperley–Lieb algebra* $\text{TL}_n(q)$ to be the quotient $\mathcal{H}_n(q)/I$, where I is the ideal generated by the element $G_{1,2}$. We have $G_{i,i+1} \in I$ for all $i = 1, \dots, n-2$, since

$$G_{i,i+1} = (G_1 G_2 \cdots G_{n-1})^{i-1} G_{1,2} (G_1 G_2 \cdots G_{n-1})^{-(i-1)}.$$

2.3. Combinatorics of partitions. Let $\lambda \vdash n$ be a partition of n , that is, $\lambda = (\lambda_1, \dots, \lambda_k)$ is a string of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ and $|\lambda| := \lambda_1 + \cdots + \lambda_k = n$. We shall also say that λ is a partition of size n .

We identify partitions with their Young diagrams: the Young diagram of λ is a left-justified array of k rows such that the j -th row contains λ_j nodes for all $j = 1, \dots, k$. We write $\theta = (x, y)$ for the node in row x and column y .

For a node θ lying in the line x and the column y of λ (that is, $\theta = (x, y)$), we define $c(\theta) := q^{y-x}$. The number $c(\theta)$ is called the (*quantum*) *content* of θ .

Now, a *tableau of shape* λ is a bijection between the set $\{1, \dots, n\}$ and the set of nodes in λ . In other words, a tableau of shape λ is obtained by placing the numbers $1, \dots, n$ in the nodes of λ . The *size* of a tableau of shape λ is n , that is, the size of λ . A tableau is *standard* if its entries increase along any row and down any column of the diagram of λ .

For a tableau \mathcal{T} , we denote by $c(\mathcal{T}|i)$ the quantum content of the node with the number i in it. For example, for the standard tableau $\mathcal{T} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$ of size 3, we have

$$c(\mathcal{T}|1) = 1, \quad c(\mathcal{T}|2) = q \quad \text{and} \quad c(\mathcal{T}|3) = q^2.$$

For any tableau \mathcal{T} of size n and any permutation $\sigma \in \mathfrak{S}_n$, we denote by \mathcal{T}^σ the tableau obtained from \mathcal{T} by applying the permutation σ on the numbers contained in the nodes of \mathcal{T} . We have

$$c(\mathcal{T}^\sigma|i) = c(\mathcal{T}|\sigma^{-1}(i)) \quad \text{for all } i = 1, \dots, n.$$

Note that if the tableau \mathcal{T} is standard, the tableau \mathcal{T}^σ is not necessarily standard.

2.4. Formulas for the irreducible representations of $\mathbb{C}(q)\mathcal{H}_n(q)$. We set $\mathbb{C}(q)\mathcal{H}_n(q) := \mathbb{C}(q) \otimes_{\mathbb{C}[q, q^{-1}]} \mathcal{H}_n(q)$. Let $\mathcal{P}(n)$ be the set of all partitions of n , and let $\lambda \in \mathcal{P}(n)$. Let V_λ be a $\mathbb{C}(q)$ -vector space with a basis $\{v_\mathcal{T}\}$ indexed by the standard tableaux of shape λ . We set $v_\mathcal{T} := 0$ for any non-standard tableau \mathcal{T} of shape λ . We have the following result on the representations of $\mathbb{C}(q)\mathcal{H}_n(q)$, established in [Ho]:

Proposition 1. *Let \mathcal{T} be a standard tableau of shape $\lambda \in \mathcal{P}(n)$. For brevity, we set $c_i := c(\mathcal{T}|i)$ for $i = 1, \dots, n$. The vector space V_λ is an irreducible representation of $\mathbb{C}(q)\mathcal{H}_n(q)$ with the action of the generators on the basis element $v_\mathcal{T}$ defined as follows: for $i = 1, \dots, n-1$,*

$$(2.3) \quad G_i(v_\mathcal{T}) = \frac{qc_{i+1} - c_{i+1}}{c_{i+1} - c_i} v_\mathcal{T} + \frac{qc_{i+1} - c_i}{c_{i+1} - c_i} v_{\mathcal{T}^{s_i}},$$

where s_i is the transposition $(i, i+1)$. Further, the set $\{V_\lambda\}_{\lambda \in \mathcal{P}(n)}$ is a complete set of pairwise non-isomorphic irreducible representations of $\mathbb{C}(q)\mathcal{H}_n(q)$.

Corollary 1. *The algebra $\mathbb{C}(q)\mathcal{H}_n(q)$ is split semisimple.*

2.5. Irreducible representations of $\mathbb{C}(q)\mathrm{TL}_n(q)$. Since the algebra $\mathbb{C}(q)\mathcal{H}_n(q)$ is semisimple, the algebra $\mathbb{C}(q)\mathrm{TL}_n(q) := \mathbb{C}(q) \otimes_{\mathbb{C}[q, q^{-1}]} \mathrm{TL}_n(q)$ is also semisimple. Moreover, the irreducible representations of $\mathbb{C}(q)\mathrm{TL}_n(q)$ are precisely the irreducible representations of $\mathbb{C}(q)\mathcal{H}_n(q)$ that pass to the quotient. That is, V_λ is an irreducible representation of $\mathbb{C}(q)\mathrm{TL}_n(q)$ if and only if $G_{1,2}(v_\mathcal{T}) = 0$ for every standard tableau \mathcal{T} of shape λ . It is easy to check that the latter is equivalent to the trivial representation not being a direct summand of the restriction $\mathrm{Res}_{\mathfrak{S}_{(s_1, s_2)}^n}^{\mathfrak{S}_n}(E^\lambda)$, where E^λ is the irreducible representation of the symmetric group \mathfrak{S}_n (equivalently, the algebra $\mathbb{C}\mathcal{H}_n(1)$) labelled by λ . We thus obtain the following description of the irreducible representations of $\mathbb{C}(q)\mathrm{TL}_n(q)$:

Proposition 2. *We have that V_λ is an irreducible representation of $\mathbb{C}(q)\mathrm{TL}_n(q)$ if and only if the Young diagram of λ has at most two columns.*

2.6. The dimension of $\mathbb{C}(q)\mathrm{TL}_n(q)$. For $n \in \mathbb{N}$, we denote by C_n the n -th Catalan number, that is, the number

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{1}{n+1} \sum_{k=0}^n \binom{n}{k}^2.$$

We have the following standard result on the dimension of $\mathbb{C}(q)\mathrm{TL}_n(q)$ (cf. [Jo1, Jo2]):

Proposition 3. *We have*

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{TL}_n(q)) = C_n.$$

3. THE FRAMISATION OF THE TEMPERLEY–LIEB ALGEBRA

In this section, we will look at a generalisation of the Temperley–Lieb algebra, which is obtained as a quotient of the Yokonuma–Hecke algebra of type A . This algebra was introduced in [GJKL2], where some of its topological properties were studied. Here we will determine its irreducible representations and calculate its dimension.

3.1. The Yokonuma–Hecke algebra $Y_{d,n}(q)$. Let $d, n \in \mathbb{N}$. Let q be an indeterminate. The *Yokonuma–Hecke algebra of type A* , denoted by $Y_{d,n}(q)$, is a $\mathbb{C}[q, q^{-1}]$ -associative algebra generated by the elements

$$g_1, \dots, g_{n-1}, t_1, \dots, t_n$$

subject to the following relations:

$$(3.1) \quad \begin{array}{lll} (b_1) & g_i g_j & = g_j g_i & \text{for all } i, j = 1, \dots, n-1 \text{ with } |i-j| > 1, \\ (b_2) & g_i g_{i+1} g_i & = g_{i+1} g_i g_{i+1} & \text{for all } i = 1, \dots, n-2, \\ (f_1) & t_i t_j & = t_j t_i & \text{for all } i, j = 1, \dots, n, \\ (f_2) & t_j g_i & = g_i t_{s_i(j)} & \text{for all } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n, \\ (f_3) & t_j^d & = 1 & \text{for all } j = 1, \dots, n, \end{array}$$

where s_i is the transposition $(i, i+1)$, together with the quadratic relations:

$$(3.2) \quad g_i^2 = q + (q-1) e_i g_i \quad \text{for all } i = 1, \dots, n-1,$$

where

$$(3.3) \quad e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}.$$

Note that we have $e_i^2 = e_i$ and $e_i g_i = g_i e_i$ for all $i = 1, \dots, n-1$. Moreover, we have

$$(3.4) \quad t_i e_i = t_{i+1} e_i \quad \text{for all } i = 1, \dots, n-1.$$

Remark 2. In [ChPA], the first author and Poulain d'Andecy consider the braid generators $\tilde{g}_i := q^{-1/2}g_i$ which satisfy the quadratic relation

$$(3.5) \quad \tilde{g}_i^2 = 1 + (q^{1/2} - q^{-1/2})e_i\tilde{g}_i.$$

On the other hand, in all the papers [Ju2, JuLa2, JuLa3, ChLa] prior to [ChPA], the authors consider the braid generators $\bar{g}_i = \tilde{g}_i + (q^{1/2} - 1)e_i\tilde{g}_i$ (and thus, $\tilde{g}_i := \bar{g}_i + (q^{-1/2} - 1)e_i\bar{g}_i$) which satisfy the quadratic relation

$$(3.6) \quad \bar{g}_i^2 = 1 + (q - 1)e_i + (q - 1)e_i\bar{g}_i.$$

Note that

$$(3.7) \quad e_i g_i = e_i \bar{g}_i = q^{1/2} e_i \tilde{g}_i \quad \text{for all } i = 1, \dots, n - 1.$$

Remark 3. If we specialise q to 1, the defining relations (3.1)–(3.2) become the defining relations for the complex reflection group $G(d, 1, n) \cong (\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$. Thus the algebra $Y_{d,n}(q)$ is a deformation of the group algebra over \mathbb{C} of the complex reflection group $G(d, 1, n)$. Moreover, for $d = 1$, the Yokonuma–Hecke algebra $Y_{1,n}(q)$ coincides with the Iwahori–Hecke algebra $\mathcal{H}_n(q)$ of type A .

Remark 4. The relations (b_1) , (b_2) , (f_1) and (f_2) are defining relations for the classical framed braid group $\mathcal{F}_n \cong \mathbb{Z} \wr B_n$, where B_n is the classical braid group on n strands, with the t_j 's being interpreted as the “elementary framings” (framing 1 on the j th strand). The relations $t_j^d = 1$ mean that the framing of each braid strand is regarded modulo d . Thus, the algebra $Y_{d,n}(q)$ arises naturally as a quotient of the framed braid group algebra over the modular relations (f_3) and the quadratic relations (3.2). Moreover, relations (3.1) are defining relations for the modular framed braid group $\mathcal{F}_{d,n} \cong (\mathbb{Z}/d\mathbb{Z}) \wr B_n$, so the algebra $Y_{d,n}(q)$ can be also seen as a quotient of the modular framed braid group algebra over the quadratic relations (3.2).

3.2. The Framisation of the Temperley–Lieb algebra $\text{FTL}_{d,n}(q)$. Let $i = 1, \dots, n - 1$. We set

$$g_{i,i+1} := 1 + g_i + g_{i+1} + g_i g_{i+1} + g_{i+1} g_i + g_i g_{i+1} g_i.$$

We define the *Framisation of the Temperley–Lieb algebra* to be the quotient $Y_{d,n}(q)/I$, where I is the ideal generated by the element

$$e_1 e_2 g_{1,2}.$$

Note that, due to (3.4), $e_1 e_2$ commutes with g_1 and with g_2 , so it commutes with $g_{1,2}$. Further, we have $e_i e_{i+1} g_{i,i+1} \in I$ for all $i = 1, \dots, n - 2$, since

$$e_i e_{i+1} g_{i,i+1} = (g_1 g_2 \dots g_{n-1})^{i-1} e_1 e_2 g_{1,2} (g_1 g_2 \dots g_{n-1})^{-(i-1)}.$$

Remark 5. In [GJKL2], the Framisation of the Temperley–Lieb algebra is defined to be the quotient $Y_{d,n}(q)/J$, where J is the ideal generated by the element $e_1 e_2 \bar{g}_{1,2}$, where

$$\bar{g}_{1,2} = 1 + \bar{g}_1 + \bar{g}_2 + \bar{g}_1 \bar{g}_2 + \bar{g}_2 \bar{g}_1 + \bar{g}_1 \bar{g}_2 \bar{g}_1.$$

Due to (3.7) and the fact that the e_i 's are idempotents, we have $e_1 e_2 \bar{g}_{1,2} = e_1 e_2 g_{1,2}$, and so $I = J$.

Remark 6. For $d = 1$, the Framisation of the Temperley–Lieb algebra $\text{FTL}_{1,n}(q)$ coincides with the classical Temperley–Lieb algebra $\text{TL}_n(q)$.

3.3. Combinatorics of d -partitions. A d -partition λ , or a Young d -diagram, of size n is a d -tuple of partitions such that the total number of nodes in the associated Young diagrams is equal to n . That is, we have $\lambda = (\lambda^{(1)}, \dots, \lambda^{(d)})$ with $\lambda^{(1)}, \dots, \lambda^{(d)}$ usual partitions such that $|\lambda^{(1)}| + \dots + |\lambda^{(d)}| = n$.

A pair $\theta = (\theta, k)$ consisting of a node θ and an integer $k \in \{1, \dots, d\}$ is called a d -node. The integer k is called the *position* of θ . A d -partition is then a set of d -nodes such that the subset consisting of the d -nodes having position k forms a usual partition, for any $k \in \{1, \dots, d\}$.

For a d -node θ lying in the line x and the column y of the k -th diagram of λ (that is, $\theta = (x, y, k)$), we define $p(\theta) := k$ and $c(\theta) := q^{y-x}$. The number $p(\theta)$ is the position of θ and the number $c(\theta)$ is called the (*quantum*) *content* of θ .

Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(d)})$ be a d -partition of n . A d -tableau of shape λ is a bijection between the set $\{1, \dots, n\}$ and the set of d -nodes in λ . In other words, a d -tableau of shape λ is obtained by placing the numbers $1, \dots, n$ in the d -nodes of λ . The *size* of a d -tableau of shape λ is n , that is, the size of λ . A d -tableau is *standard* if its entries increase along any row and down any column of every diagram in λ . For $d = 1$, a standard 1-tableau is a usual standard tableau.

For a d -tableau \mathcal{T} , we denote respectively by $p(\mathcal{T}|i)$ and $c(\mathcal{T}|i)$ the position and the quantum content of the d -node with the number i in it. For example, for the standard 3-tableau $\mathcal{T} = (\boxed{1} \boxed{3}, \emptyset, \boxed{2})$ of size 3, we have

$$p(\mathcal{T}|1) = 1, \quad p(\mathcal{T}|2) = 3, \quad p(\mathcal{T}|3) = 1 \quad \text{and} \quad c(\mathcal{T}|1) = 1, \quad c(\mathcal{T}|2) = 1, \quad c(\mathcal{T}|3) = q.$$

For any d -tableau \mathcal{T} of size n and any permutation $\sigma \in \mathfrak{S}_n$, we denote by \mathcal{T}^σ the d -tableau obtained from \mathcal{T} by applying the permutation σ on the numbers contained in the d -nodes of \mathcal{T} . We have

$$p(\mathcal{T}^\sigma|i) = p(\mathcal{T}|\sigma^{-1}(i)) \quad \text{and} \quad c(\mathcal{T}^\sigma|i) = c(\mathcal{T}|\sigma^{-1}(i)) \quad \text{for all } i = 1, \dots, n.$$

Note that if the d -tableau \mathcal{T} is standard, the d -tableau \mathcal{T}^σ is not necessarily standard.

3.4. Formulas for the irreducible representations of $\mathbb{C}(q)Y_{d,n}(q)$. The representation theory of $Y_{d,n}(q)$ has been first studied by Thiem in [Th1, Th2, Th3] and subsequently in [ChPA], where a description of its irreducible representations in terms of d -partitions and d -tableaux is given.

Let $\mathcal{P}(d, n)$ be the set of all d -partitions of n , and let $\lambda \in \mathcal{P}(d, n)$. Let \tilde{V}_λ be a $\mathbb{C}(q^{1/2})$ -vector space with a basis $\{\tilde{\mathbf{v}}_\mathcal{T}\}$ indexed by the standard d -tableaux of shape λ . In [ChPA, Proposition 5], the authors describe actions of the generators \tilde{g}_i , for $i = 1, \dots, n-1$, and t_j , for $j = 1, \dots, n$, on $\{\tilde{\mathbf{v}}_\mathcal{T}\}$, which make \tilde{V}_λ into a representation of $Y_{d,n}(q)$ over $\mathbb{C}(q^{1/2})$. The matrices describing the action of the generators t_j have complex coefficients, while the ones describing the action of the generators \tilde{g}_i have coefficients in $\mathbb{C}(q^{1/2})$. However, the change of basis

$$(3.8) \quad \mathbf{v}_\mathcal{T} := q^{N_\mathcal{T}/2} \tilde{\mathbf{v}}_\mathcal{T},$$

where $N_\mathcal{T} := \#\{i \in \{1, \dots, n-1\} \mid p(\mathcal{T}|i) < p(\mathcal{T}|i+1)\}$, and the change of generators

$$(3.9) \quad g_i = q^{1/2} \tilde{g}_i$$

yield a description of the action of $Y_{d,n}(q)$ on \tilde{V}_λ which is realised over $\mathbb{C}(q)$ (see proposition below).

Let V_λ be a $\mathbb{C}(q)$ -vector space with a basis $\{\mathbf{v}_\mathcal{T}\}$ indexed by the standard d -tableaux of shape λ . We set $\mathbf{v}_\mathcal{T} := 0$ for any non-standard d -tableau \mathcal{T} of shape λ . Let $\{\xi_1, \dots, \xi_d\}$ be

the set of all d -th roots of unity (ordered arbitrarily). We set $\mathbb{C}(q)Y_{d,n}(q) := \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} Y_{d,n}(q)$. The following result is [ChPA, Proposition 5] and [ChPA, Theorem 1], with the change of basis and generators described by (3.8) and (3.9).

Proposition 4. *Let \mathcal{T} be a standard d -tableau of shape $\lambda \in \mathcal{P}(d, n)$. For brevity, we set $p_i := p(\mathcal{T}|i)$ and $c_i := c(\mathcal{T}|i)$ for $i = 1, \dots, n$. The vector space V_λ is an irreducible representation of $\mathbb{C}(q)Y_{d,n}(q)$ with the action of the generators on the basis element $\mathbf{v}_\mathcal{T}$ defined as follows: for $j = 1, \dots, n$,*

$$(3.10) \quad t_j(\mathbf{v}_\mathcal{T}) = \xi_{p_j} \mathbf{v}_\mathcal{T} ;$$

for $i = 1, \dots, n-1$, if $p_i > p_{i+1}$ then

$$(3.11) \quad g_i(\mathbf{v}_\mathcal{T}) = \mathbf{v}_{\mathcal{T}^{s_i}} ,$$

if $p_i < p_{i+1}$ then

$$(3.12) \quad g_i(\mathbf{v}_\mathcal{T}) = q \mathbf{v}_{\mathcal{T}^{s_i}} ,$$

and if $p_i = p_{i+1}$ then

$$(3.13) \quad g_i(\mathbf{v}_\mathcal{T}) = \frac{q^{c_{i+1}} - c_{i+1}}{c_{i+1} - c_i} \mathbf{v}_\mathcal{T} + \frac{q^{c_i} - c_i}{c_{i+1} - c_i} \mathbf{v}_{\mathcal{T}^{s_i}} ,$$

where s_i is the transposition $(i, i+1)$. Further, the set $\{V_\lambda\}_{\lambda \in \mathcal{P}(d,n)}$ is a complete set of pairwise non-isomorphic irreducible representations of $\mathbb{C}(q)Y_{d,n}(q)$.

Corollary 2. *The algebra $\mathbb{C}(q)Y_{d,n}(q)$ is split semisimple.*

Remark 7. Note that

$$(3.14) \quad e_i(\mathbf{v}_\mathcal{T}) = \begin{cases} 0 & , \text{ if } p_i \neq p_{i+1}; \\ \mathbf{v}_\mathcal{T} & , \text{ if } p_i = p_{i+1}. \end{cases}$$

3.5. Irreducible representations of $\mathbb{C}(q)\text{FTL}_{d,n}(q)$. Since the algebra $\mathbb{C}(q)Y_{d,n}(q)$ is semisimple, the algebra $\mathbb{C}(q)\text{FTL}_{d,n}(q) := \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} \text{FTL}_{d,n}(q)$ is also semisimple. Moreover, the irreducible representations of $\mathbb{C}(q)\text{FTL}_{d,n}(q)$ are precisely the irreducible representations of $\mathbb{C}(q)Y_{d,n}(q)$ that pass to the quotient. That is, V_λ is an irreducible representation of $\mathbb{C}(q)\text{FTL}_{d,n}(q)$ if and only if $e_1 e_2 g_{1,2}(\mathbf{v}_\mathcal{T}) = 0$ for every standard d -tableau \mathcal{T} of shape λ .

Proposition 5. *We have that V_λ is an irreducible representation of $\mathbb{C}(q)\text{FTL}_{d,n}(q)$ if and only if the Young diagram of $\lambda^{(i)}$ has at most two columns for all $i = 1, \dots, d$.*

Proof. Let us assume first that V_λ is an irreducible representation of $\mathbb{C}(q)\text{FTL}_{d,n}(q)$ and let $i \in \{1, \dots, d\}$. Set $n_i := |\lambda^{(i)}|$. If $n_i \leq 2$, then $\lambda^{(i)}$ has at most two columns. If $n_i \geq 3$, let us consider all the standard d -tableaux $\mathcal{T} = (\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(d)})$ of shape λ such that

$$p_1 = p_2 = p_3 = \dots = p_{n_i} = i.$$

Then, using the notation of Proposition 1 for the Iwahori–Hecke algebra $\mathcal{H}_{n_i}(q)$ and Equation (3.14), we obtain

$$G_{1,2}(\mathbf{v}_{\mathcal{T}^{(i)}}) = g_{1,2}(\mathbf{v}_\mathcal{T}) = g_{1,2} e_1 e_2(\mathbf{v}_\mathcal{T}) = e_1 e_2 g_{1,2}(\mathbf{v}_\mathcal{T}) = 0$$

Since $\mathcal{T}^{(i)}$ runs over all the standard tableaux of shape $\lambda^{(i)}$, Proposition 2 yields that $\lambda^{(i)}$ has at most two columns.

Now assume that the Young diagram of $\lambda^{(i)}$ has at most two columns for all $i = 1, \dots, d$. Let $\mathcal{T} = (\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(d)})$ be a standard d -tableau of shape λ . If $p_1 = p_2 =$

$p_3 =: p$, then, by (3.14), $e_1 e_2 g_{1,2}(\mathbf{v}_\tau) = g_{1,2} e_1 e_2(\mathbf{v}_\tau) = g_{1,2}(\mathbf{v}_\tau)$. In this case, $g_{1,2}$ acts on \mathbf{v}_τ in the same way that $G_{1,2}$ acts on $v_{\tau^{(p)}}$ (replacing the entries greater than 3 by entries in $\{4, \dots, |\lambda^{(p)}|\}$). Following the result on the classical case, we have $g_{1,2}(\mathbf{v}_\tau) = 0$. Otherwise, again by (3.14), we have $e_1 e_2(\mathbf{v}_\tau) = 0$, and $e_1 e_2 g_{1,2}(\mathbf{v}_\tau) = g_{1,2} e_1 e_2(\mathbf{v}_\tau) = 0$ as desired. \square

3.6. The dimension of $\mathbb{C}(q)\text{FTL}_{d,n}(q)$. We will now use the complete description of the irreducible representations of $\mathbb{C}(q)\text{FTL}_{d,n}(q)$ by Proposition 5 to obtain a dimension formula for $\mathbb{C}(q)\text{FTL}_{d,n}(q)$. Set

$$\mathcal{K}_{d,n} := \{(k_1, k_2, \dots, k_d) \in \mathbb{N}^d \mid k_1 + k_2 + \dots + k_d = n\}.$$

Proposition 6. *We have*

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\text{FTL}_{d,n}(q)) = \sum_{(k_1, k_2, \dots, k_d) \in \mathcal{K}_{d,n}} \left(\frac{n!}{k_1! k_2! \dots k_d!} \right)^2 C_{k_1} C_{k_2} \dots C_{k_d}.$$

Proof. Let us denote by $\mathcal{P}^{\leq 2}(d, n)$ the set of d -partitions λ of n such that the Young diagram of $\lambda^{(i)}$ has at most two columns for all $i = 1, \dots, d$. By Proposition 5, and since the algebra $\mathbb{C}(q)\text{FTL}_{d,n}(q)$ is semisimple, we have

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\text{FTL}_{d,n}(q)) = \sum_{\lambda \in \mathcal{P}^{\leq 2}(d, n)} \dim_{\mathbb{C}(q)}(V_\lambda)^2,$$

where $\dim_{\mathbb{C}(q)}(V_\lambda)$ is the number of standard d -tableaux of shape λ .

Fix $(k_1, k_2, \dots, k_d) \in \mathcal{K}_{d,n}$. We denote by $\mathcal{P}^{\leq 2}(k_1, k_2, \dots, k_d)$ the set of all d -partitions λ in $\mathcal{P}^{\leq 2}(d, n)$ such that $|\lambda^{(i)}| = k_i$ for all $i = 1, \dots, d$. We have

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\text{FTL}_{d,n}(q)) = \sum_{(k_1, k_2, \dots, k_d) \in \mathcal{K}_{d,n}} \sum_{\lambda \in \mathcal{P}^{\leq 2}(k_1, k_2, \dots, k_d)} \dim_{\mathbb{C}(q)}(V_\lambda)^2.$$

Let $\lambda \in \mathcal{P}^{\leq 2}(k_1, k_2, \dots, k_d)$. We have

$$\binom{n}{k_1} \binom{n-k_1}{k_2} \binom{n-k_1-k_2}{k_3} \dots \binom{n-k_1-k_2-\dots-k_{d-1}}{k_d} = \frac{n!}{k_1! k_2! \dots k_d!}$$

ways to choose the numbers in $\{1, \dots, n\}$ that will be placed in the nodes of the Young diagram of $\lambda^{(i)}$ for each $i = 1, \dots, d$. We deduce that

$$\dim_{\mathbb{C}(q)}(V_\lambda) = \frac{n!}{k_1! k_2! \dots k_d!} \prod_{i=1}^d \dim_{\mathbb{C}(q)}(V_{\lambda^{(i)}}),$$

where $V_{\lambda^{(i)}}$ is the irreducible representation of $\mathbb{C}(q)\text{TL}_{k_i}(q)$ labelled by $\lambda^{(i)}$. We thus obtain that $\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\text{FTL}_{d,n}(q))$ is equal to

$$\sum_{(k_1, k_2, \dots, k_d) \in \mathcal{K}_{d,n}} \left(\frac{n!}{k_1! k_2! \dots k_d!} \right)^2 \sum_{\lambda \in \mathcal{P}^{\leq 2}(k_1, k_2, \dots, k_d)} \prod_{i=1}^d \dim_{\mathbb{C}(q)}(V_{\lambda^{(i)}})^2.$$

We now have that

$$\sum_{\lambda \in \mathcal{P}^{\leq 2}(k_1, k_2, \dots, k_d)} \prod_{i=1}^d \dim_{\mathbb{C}(q)}(V_{\lambda^{(i)}})^2$$

is equal to

$$\sum_{\lambda^{(1)} \in \mathcal{P}^{\leq 2}(1, k_1)} \sum_{\lambda^{(2)} \in \mathcal{P}^{\leq 2}(1, k_2)} \cdots \sum_{\lambda^{(d)} \in \mathcal{P}^{\leq 2}(1, k_d)} \prod_{i=1}^d \dim_{\mathbb{C}(q)}(V_{\lambda^{(i)}})^2,$$

which in turn is equal to

$$\prod_{i=1}^d \left(\sum_{\lambda^{(i)} \in \mathcal{P}^{\leq 2}(1, k_i)} \dim_{\mathbb{C}(q)}(V_{\lambda^{(i)}})^2 \right).$$

By Proposition 3, we have that

$$\sum_{\lambda^{(i)} \in \mathcal{P}^{\leq 2}(1, k_i)} \dim_{\mathbb{C}(q)}(V_{\lambda^{(i)}})^2 = \dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{TL}_{k_i}(q)) = C_{k_i},$$

for all $i = 1, \dots, d$. We conclude that

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)) = \sum_{(k_1, k_2, \dots, k_d) \in \mathcal{K}_{d,n}} \left(\frac{n!}{k_1! k_2! \cdots k_d!} \right)^2 C_{k_1} C_{k_2} \cdots C_{k_d}.$$

□

REFERENCES

- [CJKL] M. Chlouveraki, J. Juyumaya, K. Karvounis, S. Lambropoulou, *Identifying the invariants for classical knots and links from the Yokonuma–Hecke algebras*, arXiv:1505.06666.
- [ChLa] M. Chlouveraki, S. Lambropoulou, *The Yokonuma–Hecke algebras and the HOMFLYPT polynomial*, J. Knot Theory Ramifications **22**, No. 14 (2013), 1350080.
- [ChPo] M. Chlouveraki, G. Pouchin, *Determination of the representations and a basis for the Yokonuma–Temperley–Lieb algebra*, Algebras and Representation Theory **18**(2) (2015), 421–447.
- [ChPA] M. Chlouveraki, L. Poulain d’Andecy, *Representation theory of the Yokonuma–Hecke algebra*, Adv. Math. **259** (2014), 134–172.
- [Ho] P. Hoefsmit, *Representations of Hecke algebras of finite groups with BN-pairs of classical type*, Ph. D. thesis, University of British Columbia, 1974.
- [GJKL1] D. Goundaroulis, J. Juyumaya, A. Kontogeorgis, S. Lambropoulou, *The Yokonuma–Temperley–Lieb algebra*, Banach Center Publications, Vol. 103, pg. 24, 2014.
- [GJKL2] D. Goundaroulis, J. Juyumaya, A. Kontogeorgis, S. Lambropoulou, *Framization of the Temperley–Lieb algebra*, arXiv:1304.7440.
- [Jo1] V. F. R. Jones, *Index for Subfactors*, Invent. Math. **72** (1983), 1–25.
- [Jo2] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Annals of Math. **126** (1987), no. 2, 335–388.
- [Ju1] J. Juyumaya, *Sur les nouveaux générateurs de l’algèbre de Hecke $H(G, U, 1)$* , Journal of Algebra **204** (1998) 49–68.
- [JuKa] J. Juyumaya, S. Kannan, *Braid relations in the Yokonuma–Hecke algebra*, Journal of Algebra **239** (2001) 272–297.
- [Ju2] J. Juyumaya, *Markov trace on the Yokonuma–Hecke algebra*, J. Knot Theory and its Ramifications **13** (2004) 25–39.
- [JuLa1] J. Juyumaya, S. Lambropoulou, *p -adic framed braids*, Topology and its Applications **154** (2007) 1804–1826.
- [JuLa2] J. Juyumaya, S. Lambropoulou, *p -adic framed braids II*, Adv. Math. **234** (2013) 149–191.
- [JuLa3] J. Juyumaya, S. Lambropoulou, *An adelic extension of the Jones polynomial*, M. Banagl, D. Vogel (eds.) The mathematics of knots, Contributions in the Mathematical and Computational Sciences, Vol. 1, Springer, 2011.
- [JuLa4] J. Juyumaya, S. Lambropoulou, *An invariant for singular knots*, J. Knot Theory and its Ramifications, **18**(6) (2009) 825–840.
- [JuLa5] J. Juyumaya, S. Lambropoulou, *On the framization of knot algebras*, to appear in New Ideas in Low-dimensional Topology, L. Kauffman, V. Manturov (eds), Series of Knots and Everything, WS.

- [TeLi] N. Temperley, E. Lieb, *Relations between the 'Percolation' and 'Colouring' Problem and other Graph-Theoretical Problems Associated with Regular Planar Lattices: Some Exact Results for the 'Percolation' Problem*, Proceedings of the Royal Society Series A **322** (1971) 251–280.
- [Th1] N. Thiem, *Unipotent Hecke algebras: the structure, representation theory, and combinatorics*, Ph.D Thesis, University of Wisconsin (2004).
- [Th2] N. Thiem, *Unipotent Hecke algebras of $GL_n(\mathbb{F}_q)$* , Journal of Algebra **284** (2005) 559–577.
- [Th3] N. Thiem, *A skein-like multiplication algorithm for unipotent Hecke algebras*, Trans. Amer. Math. Soc. **359**(4) (2007) 1685–1724.
- [Yo] T. Yokonuma, *Sur la structure des anneaux de Hecke d'un groupe de Chevalley fini*, C.R. Acad. Sc. Paris **264** (1967) 344–347.