# Blocks and Schur elements for Hecke algebras of exceptional complex reflection groups

**Maria Chlouveraki**

*National and Kapodistrian University of Athens*

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Let $W \subset \mathrm{GL}(V)$ be an irreducible complex reflection group (i.e., $W$ acts irreducibly on $V$). Then one of the following assertions is true:

- $W \cong G(de, e, r)$, where $G(de, e, r)$ is the group of all $r \times r$ monomial matrices whose non-zero entries are $de$-th roots of unity, while the product of all non-zero entries is a $d$-th root of unity.
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**Remark:** The groups $G_4, G_5, \ldots, G_{22}$ are of rank 2.

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It has been proved for :

- the real reflection groups by Bourbaki;
- the complex reflection groups $G(de, e, r)$ by Ariki–Koike, Broué–Malle, Ariki;
- the group $G_4$ by Broué–Malle, Funar, Marin, Chavli;
- the groups $G_5, \ldots, G_{16}$ by Chavli ($G_{12}$ also by Marin–Pfeiffer);
- the groups $G_{17}$, $G_{18}$, $G_{19}$ by Tsuchioka;
- the groups $G_{20}$, $G_{21}$ by Marin;
- the groups $G_{22}, \ldots, G_{37}$ by Marin, Marin–Pfeiffer.

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There exists a linear map $\boldsymbol{\tau} : \mathcal{H}(W) \to R_W$ that satisfies:

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- the real reflection groups by Bourbaki;
- the complex reflection groups $G(de, e, r)$ by Bremke–Malle, Malle–Mathas;
- the groups $G_4$, $G_{12}$, $G_{22}$, $G_{24}$ by Malle–Michel ($G_4$ also by Marin–Wagner);
- the groups $G_4$, $G_5$, $G_6$, $G_7$, $G_8$ by Boura–Chavli–C.–Karvounis;
- the group $G_{13}$ by Boura–Chavli–C.;
- the groups $G_4$, $G_5$, $\ldots$, $G_{15}$ by Chavli–Pfeiffer.

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## Difficulties

- Not any basis will work.
- Step 3 is easier said than done!

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**GAP3 - Chevie package:**

1. SchurElements(H);
2. FactorizedSchurElements(H);

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$$D_\theta = \underbrace{\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}}_{\mathrm{Irr}(\mathcal{H}_\theta(W))} \left.\vphantom{\begin{pmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}\right\} \mathrm{Irr}(W)$$

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**Blocks for exceptional complex reflection groups are known**

- for real reflection groups by Geck-Pfeiffer;
- for $G_4, G_5, G_8, G_9, G_{10}, G_{12}, G_{16}, G_{20}, G_{22}$ by C.-Miyachi;
- for $G_4, G_8, G_{16}$ by Chavli;
- for $G_{24}, G_{25}, G_{26}, G_{31}, G_{32}, G_{33}$ by C.-Malle.

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## Theorem (Geck-Pfeiffer '00)

We have that $\chi, \psi$ are in the same block if and only if $\theta(\omega_\chi(z)) = \theta(\omega_\psi(z))$ for all $z \in Z(\mathcal{H}(W))$, where $\omega_\chi(z) = \chi(z)/\chi(1)$.

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1. Let $B(W) = \langle \boldsymbol{\beta} \rangle$. If $\theta(\omega_\chi(\boldsymbol{\beta})) \neq \theta(\omega_\psi(\boldsymbol{\beta}))$, then $\chi, \psi$ are not in the same block.

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3. Any representation of dimension 1 is irreducible. A representation of dimension 2 is irreducible, unless it has a 1-dimensional subrepresentation. A representation of dimension 3 is irreducible unless it or its transpose has a 1-dimensional subrepresentation.
4. Let $\phi \in \mathrm{Irr}(\mathcal{H}_\theta(W))$ and let $P(q)$ be a polynomial that is divisible by $S_\chi$ for all $\chi$ such that $d_{\chi,\phi} \neq 0$. Then
$$\sum_{\chi \in \mathrm{Irr}(W)} d_{\chi,\phi} \cdot \theta(P(q)/S_\chi) = 0.$$

# The Trinh-Xue Conjecture '23

## Conjecture

For any generic finite reductive group $G$ and integers $d, e > 0$, the intersection of a $d$-Harish-Chandra series and an $e$-Harish-Chandra series of $G$ is parametrised by a union of blocks of the Hecke algebra of the $d$-cuspidal pair at an $e$-th root of unity, and similarly for the Hecke algebra of the $e$-cuspidal pair at a $d$-th root of unity. These parametrisations match the blocks on the two sides.

# The Trinh-Xue Conjecture '23

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- A generalised conjecture holds for the non-crystallographic Coxeter groups as well as for the "spetsial" complex reflection groups $G_4, G_6, G_8$ and $G_{24}$.