Generalising the Temperley-Lieb algebra: Representations & Knots

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A Markov trace on $\mathcal{H}_n(q)$ and $TL_n(q)$

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$$\tau : \mathcal{H}_n(q) \longrightarrow \mathbb{C}(q, z) \qquad \text{Markov trace (Ocneanu trace)}$$

$$1 \longmapsto 1$$

$$g_i \longmapsto z$$

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Markov trace (Ocneanu trace)

τ passes to $TL_n(q)$ for $z = -\dfrac{1}{q+1} \Leftarrow \tau(G_1) = 0$

The braid group and Alexander's Theorem

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i = 1, \ldots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| > 1 \end{array} \right\rangle$$

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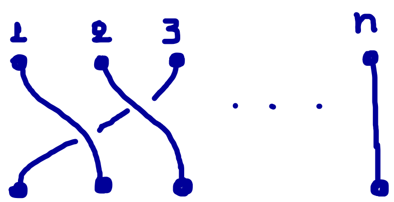

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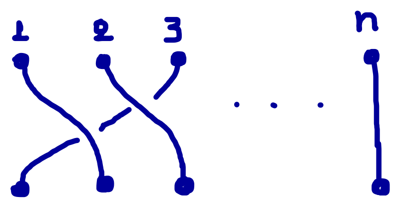

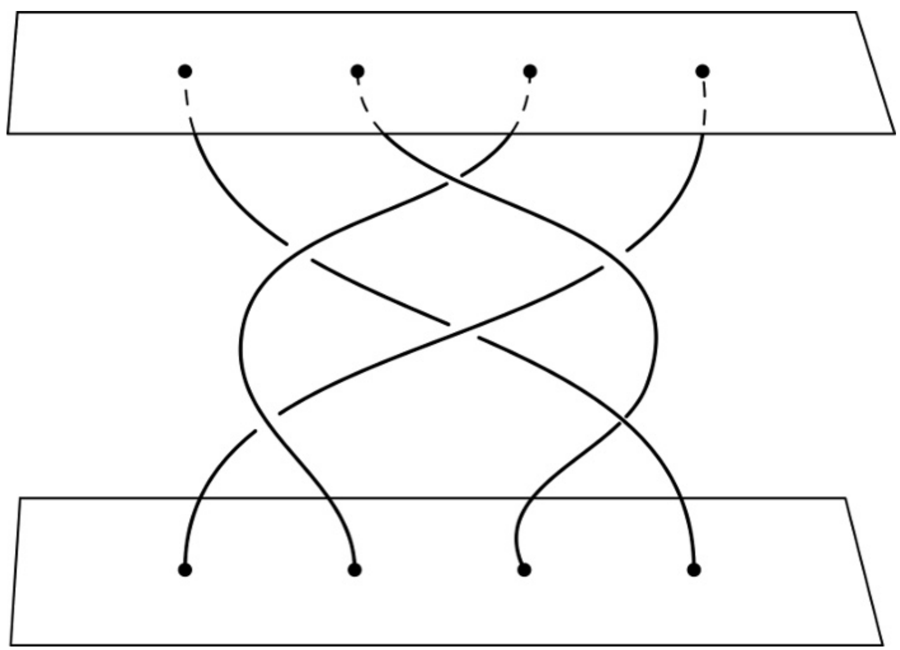

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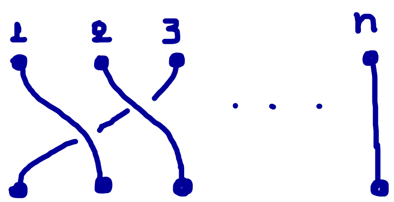
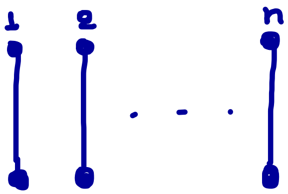
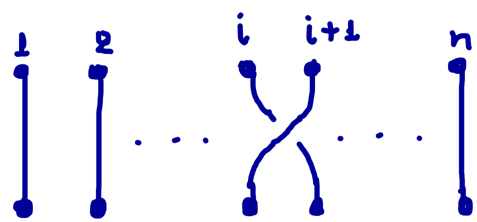
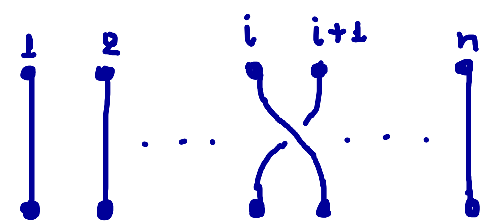


Id $=$ 

$\sigma_i =$ 

$\sigma_i^{-1} =$ 

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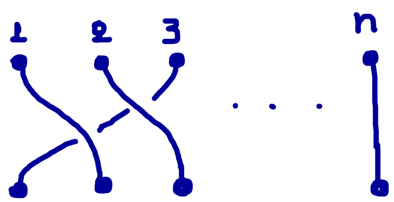
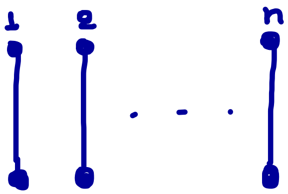
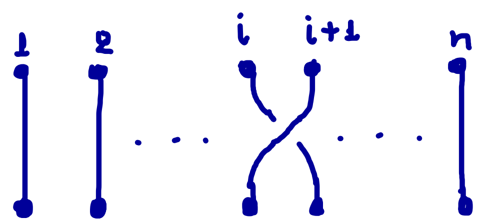
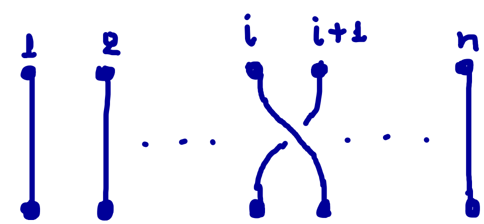
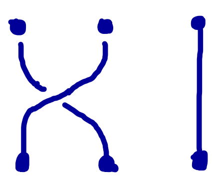


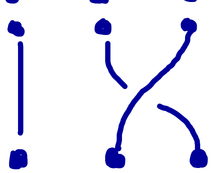
Id = 

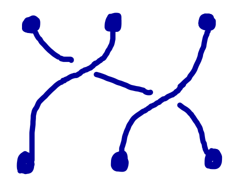
σ_i = 

σ_i^{-1} = 

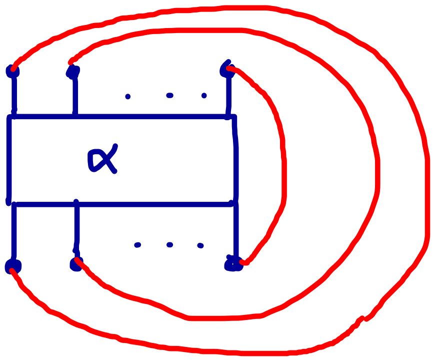
Multiplication : concatenation of diagrams

<u>Ex.</u> $\alpha = \sigma_1 =$ 

$\beta = \sigma_2 =$ 

$\Rightarrow \alpha\beta = \sigma_1\sigma_2 =$ 

The braid group and Alexander's Theorem



$=: \hat{\alpha}$ = closure of α

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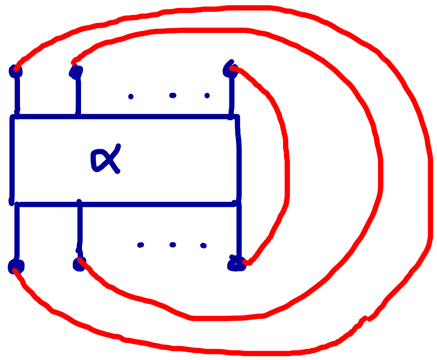

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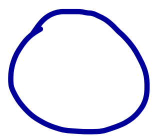
Ex. $\alpha =$  $\hat{\alpha} =$ 

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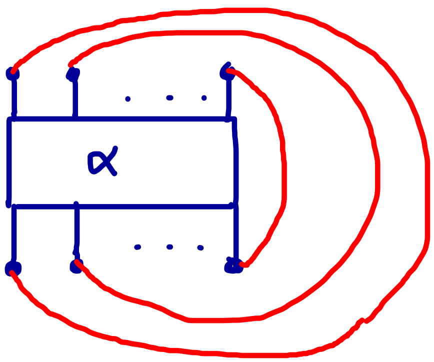

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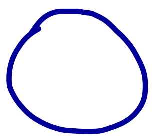
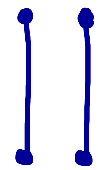
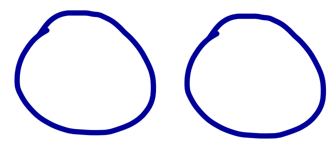
Ex. $\alpha =$  $\hat{\alpha} =$ 

$\alpha =$  $\hat{\alpha} =$ 

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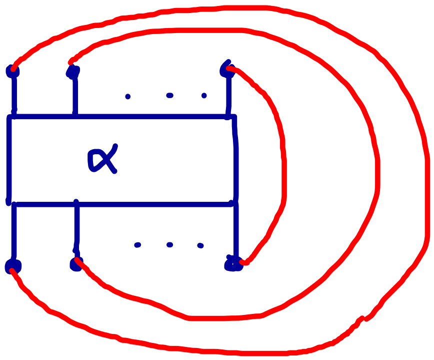

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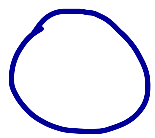
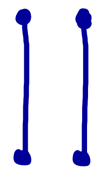
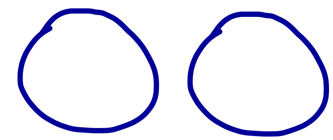
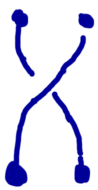
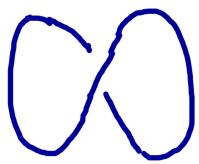
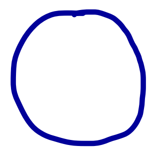
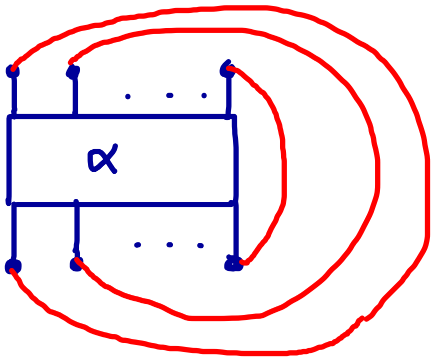
<u>Ex.</u> $\alpha =$  $\hat{\alpha} =$ 

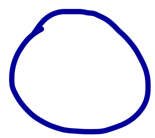
$\alpha =$  $\hat{\alpha} =$ 

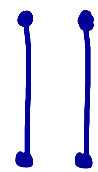
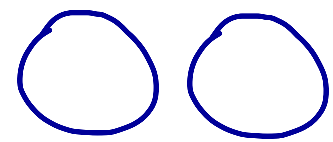
$\alpha = \sigma_1 =$  $\hat{\alpha} =$  $\sim$ 

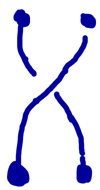
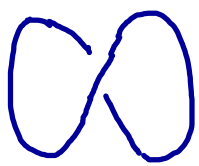
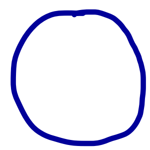
The braid group and Alexander's Theorem

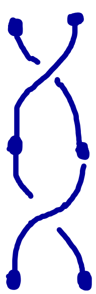
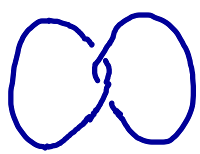
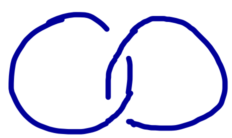


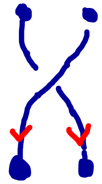
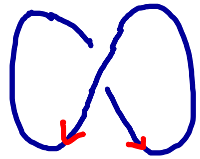
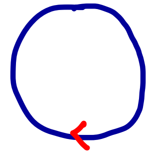
$=: \hat{\alpha} =$ closure of α

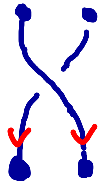
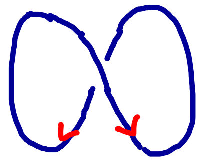
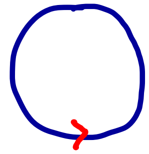
Ex. $\alpha =$  $\hat{\alpha} =$ 

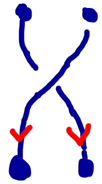
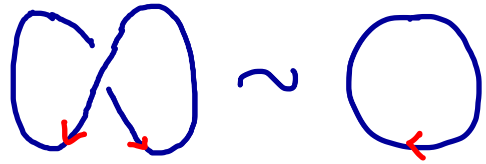
$\alpha =$  $\hat{\alpha} =$ 

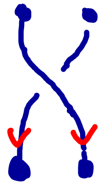
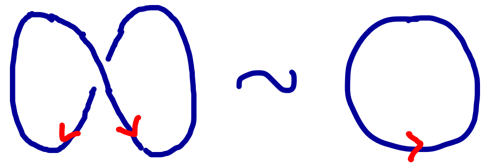
$\alpha = \sigma_1 =$  $\hat{\alpha} =$  $\sim$ 

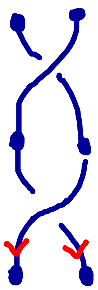
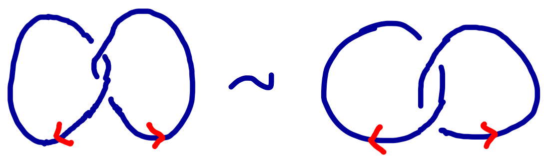
$\alpha = \sigma_1^2 =$  $\hat{\alpha} =$  $\sim$ 

$\alpha = \sigma_1 =$  $\hat{\alpha} =$  $\sim$ 

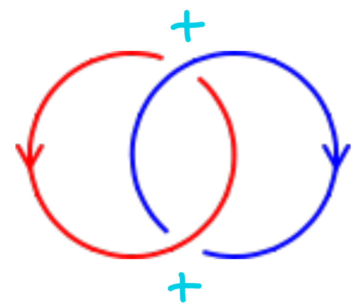
$\alpha = \sigma_1^{-1} =$  $\hat{\alpha} =$  $\sim$ 

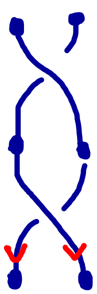
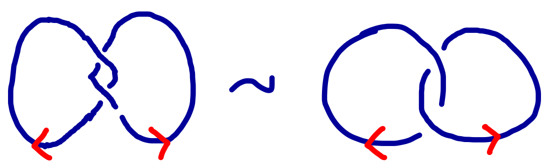
$\alpha = \sigma_1 =$  $\hat{\alpha} =$ 

$\alpha = \sigma_1^{-1} =$  $\hat{\alpha} =$ 

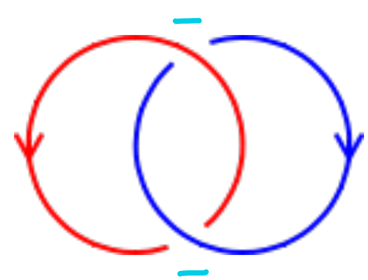
$\alpha = \sigma_1^{2} =$  $\hat{\alpha} =$ 

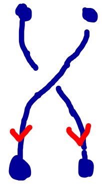
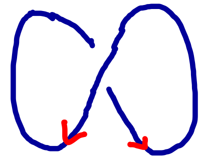
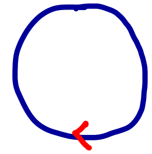
right Hopf

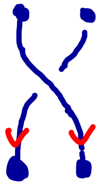
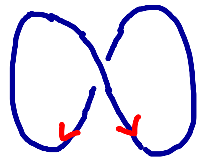
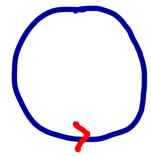


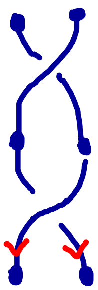
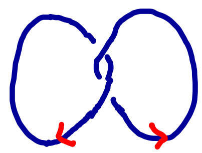
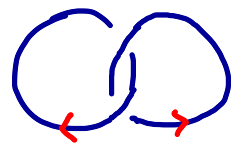
$\alpha = \sigma_1^{-2} =$  $\hat{\alpha} =$ 

left Hopf

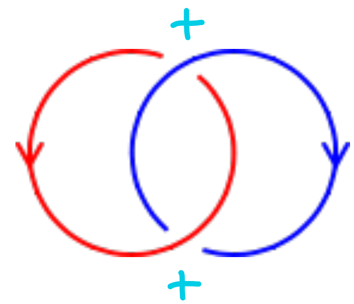


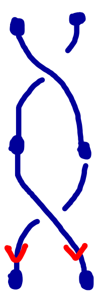
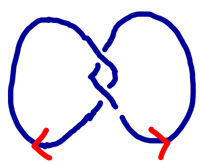
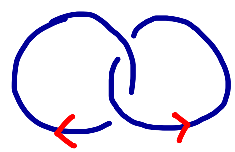
$\alpha = \sigma_1 =$  $\hat{\alpha} =$  $\sim$ 

$\alpha = \sigma_1^{-1} =$  $\hat{\alpha} =$  $\sim$ 

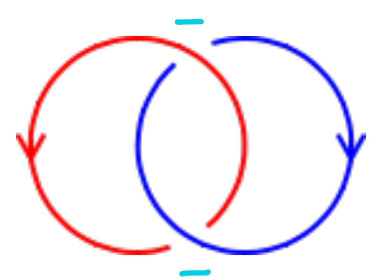
$\alpha = \sigma_1^{2} =$  $\hat{\alpha} =$  $\sim$ 

right Hopf

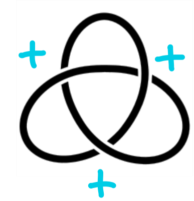


$\alpha = \sigma_1^{-2} =$  $\hat{\alpha} =$  $\sim$ 

left Hopf



$\alpha = \sigma_1^{3}$ $\hat{\alpha} =$ right-handed trefoil



$\alpha = \sigma_1^{-3}$ $\hat{\alpha} =$ left-handed trefoil

Construction of the Jones polynomial #1

$$\begin{array}{ccccccccc} B_n & \hookrightarrow & \mathbb{C}(q)[B_n] & \twoheadrightarrow & \mathcal{H}_n(q) & \twoheadrightarrow & TL_n(q) & \xrightarrow{\tau} & \mathbb{C}(q) \\ \sigma_i & \longmapsto & \sigma_i & \longmapsto & g_i & \longmapsto & g_i & & \end{array}$$

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$P : B_n \longrightarrow \mathbb{C}(q)$ is a knot invariant iff $P(\alpha\beta) = P(\beta\alpha)$ and $P(\alpha) = P(\alpha\sigma_n^{\pm 1})$

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✓

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$\alpha \longmapsto \mathcal{J}(\alpha)\tau(\alpha)$

✓

Construction of the Jones polynomial #1

$$B_n \hookrightarrow \mathbb{C}(q)[B_n] \twoheadrightarrow \mathcal{H}_n(q) \twoheadrightarrow TL_n(q) \xrightarrow{\tau} \mathbb{C}(q)$$
$$\sigma_i \longmapsto \sigma_i \longmapsto g_i \longmapsto g_i$$

$P : B_n \longrightarrow \mathbb{C}(q)$ is a knot invariant iff $P(\alpha\beta) = P(\beta\alpha)$ and $P(\alpha) = P(\alpha\sigma_n^{\pm 1})$

$\alpha \longmapsto \gamma(\alpha)\tau(\alpha)$

$\gamma(\alpha) = (x + x^{-1})^{n-1}(-x)^{\exp(\alpha)}$ $\quad x = q^{1/2}$ $\quad \exp(\alpha) = \sum r_i$ for $\alpha = \prod \sigma_i^{r_i}$

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$\checkmark$

$$\mathcal{J}(\alpha) = (x + x^{-1})^{n-1}(-x)^{\exp(\alpha)} \qquad x = q^{1/2} \qquad \exp(\alpha) = \sum r_i \text{ for } \alpha = \prod \sigma_i^{r_i}$$

<u>Ex.</u> $n = 1$: $P(1) = 1$

Construction of the Jones polynomial #1

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$P : B_n \longrightarrow \mathbb{C}(q)$ is a knot invariant iff $P(\alpha\beta) = P(\beta\alpha)$ ✓ and $P(\alpha) = P(\alpha\sigma_n^{\pm 1})$

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Ex. $n = 1$: $P(1) = 1$

$n = 2$: $P(1) = x + x^{-1}$

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$$\eta(\alpha) = (x + x^{-1})^{n-1}(-x)^{\exp(\alpha)} \qquad x = q^{1/2} \qquad \exp(\alpha) = \sum r_i \ \text{ for } \alpha = \prod \sigma_i^{r_i}$$

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$$P(\sigma_1) = (x + x^{-1})(-x)(-x^2 - 1)^{-1} = 1$$

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Construction of the Jones polynomial #1

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$$P(\sigma_1^2) = (x + x^{-1})\, x^2 \left(x^2 - \frac{x^2 - 1}{x^2 + 1} \right) = x + x^5$$

Construction of the Jones polynomial #1

$$
\begin{array}{ccccccccc}
B_n & \hookrightarrow & \mathbb{C}(q)[B_n] & \twoheadrightarrow & \mathcal{H}_n(q) & \twoheadrightarrow & TL_n(q) & \xrightarrow{\tau} & \mathbb{C}(q) \\
\sigma_i & \longmapsto & \sigma_i & \longmapsto & g_i & \longmapsto & g_i & &
\end{array}
$$

$$
\begin{array}{rcl}
P : B_n & \longrightarrow & \mathbb{C}(q) \\
\alpha & \longmapsto & \eta(\alpha)\tau(\alpha)
\end{array}
$$

is a knot invariant iff $P(\alpha\beta) = P(\beta\alpha)$ and $P(\alpha) = P(\alpha\sigma_n^{\pm 1})$

$$\eta(\alpha) = (x + x^{-1})^{n-1}(-x)^{\exp(\alpha)} \qquad x = q^{1/2} \qquad \exp(\alpha) = \sum r_i \text{ for } \alpha = \prod \sigma_i^{r_i}$$

Ex. $n = 1$: $P(1) = 1$

$n = 2$: $P(1) = x + x^{-1}$

$\tau(1) = 1$

$$P(\sigma_1) = (x + x^{-1})(-x)\underbrace{(-x^2 - 1)^{-1}}_{\tau(g_1) = z} = 1$$

$$P(\sigma_1^2) = (x + x^{-1})\, x^2 \underbrace{\left(x^2 - \frac{x^2 - 1}{x^2 + 1} \right)}_{\tau(g_1^2) = q + (q-1)z} = x + x^5$$

Construction of the Jones polynomial #2

- $P(\bigcirc) = 1$

- $$\boxed{x^{-2}\, P\Big(\underset{D^+}{\times}\Big) - x^{2}\, P\Big(\underset{D^-}{\times}\Big) = (x^{-1} - x)\, P\Big(\underset{D_0}{\downarrow\downarrow}\Big)}$$

Skein relation

Construction of the Jones polynomial #2

- $P(\bigcirc) = 1$

- $$\boxed{x^{-2}\, P(\underset{D^+}{\times}) - x^{2}\, P(\underset{D^-}{\times}) = (x^{-1}-x)\, P(\underset{D_0}{\downarrow\downarrow})}$$

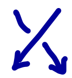
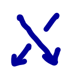
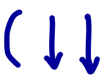
Skein relation

<u>Ex.</u> $$x^{-2}\underbrace{P(\infty)}_{=1} - x^{2}\underbrace{P(\infty)}_{=1} = (x^{-1}-x)\, P(\bigcirc\ \bigcirc)$$

$$\Rightarrow \boxed{P(\bigcirc\ \bigcirc) = \frac{x^{2}-x^{-2}}{x-x^{-1}} = x+x^{-1}}$$

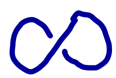
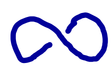
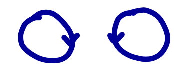
Construction of the Jones polynomial #2

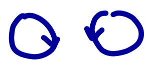
- $P($  $) = 1$

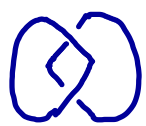
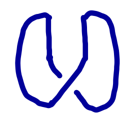
- $x^{-2}\,P($  $) \;-\; x^{2}\,P($  $) = (x^{-1}-x)\,P($  $)$

  D^+ $\qquad$ D^- $\qquad$ D_0

Skein relation

<u>Ex.</u> $x^{-2}\,\underbrace{P(}_{}$  $\underbrace{)}_{=1} \;-\; x^{2}\,\underbrace{P(}_{}$  $\underbrace{)}_{=1} = (x^{-1}-x)\,P($  $)$

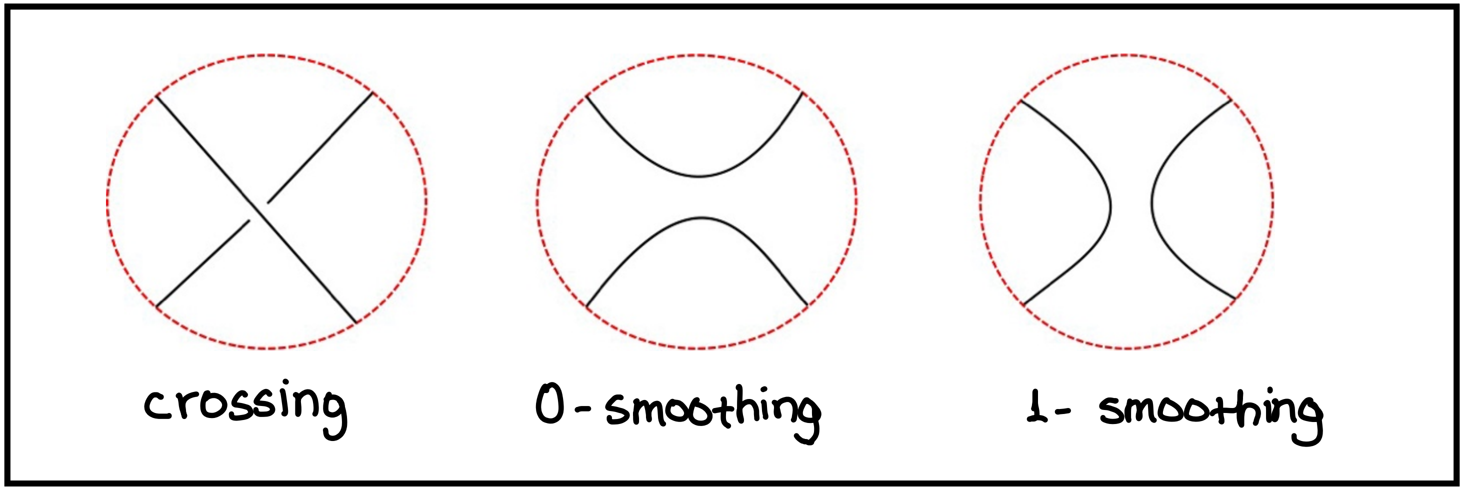
$\Longrightarrow$ $\boxed{P(}$  $\boxed{) = \dfrac{x^2-x^{-2}}{x-x^{-1}} = x+x^{-1}}$

$x^{-2}\,P($  $) \;-\; x^{2}\,\underbrace{P(}$  $\underbrace{)}_{P(\text{○○})}$ $= (x^{-1}-x)\,\underbrace{P(}$  $\underbrace{)}_{=1}$

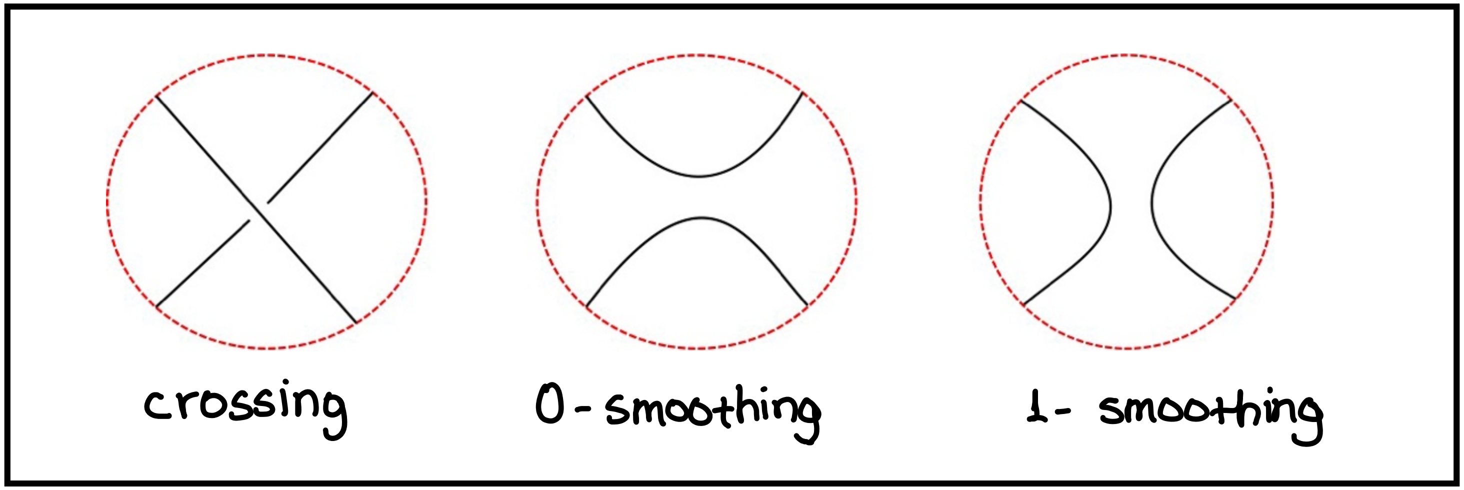
$P($  $)$

$\Longrightarrow$ $\boxed{P(}$  $\boxed{) = x^2\left[(x^{-1}-x) + x^2(x+x^{-1})\right] = x^2(x^{-1}+x^3) = x+x^5}$

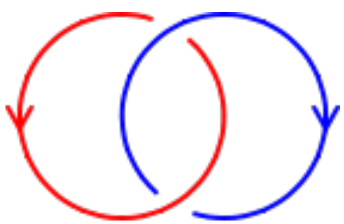
Construction of the Jones polynomial #3

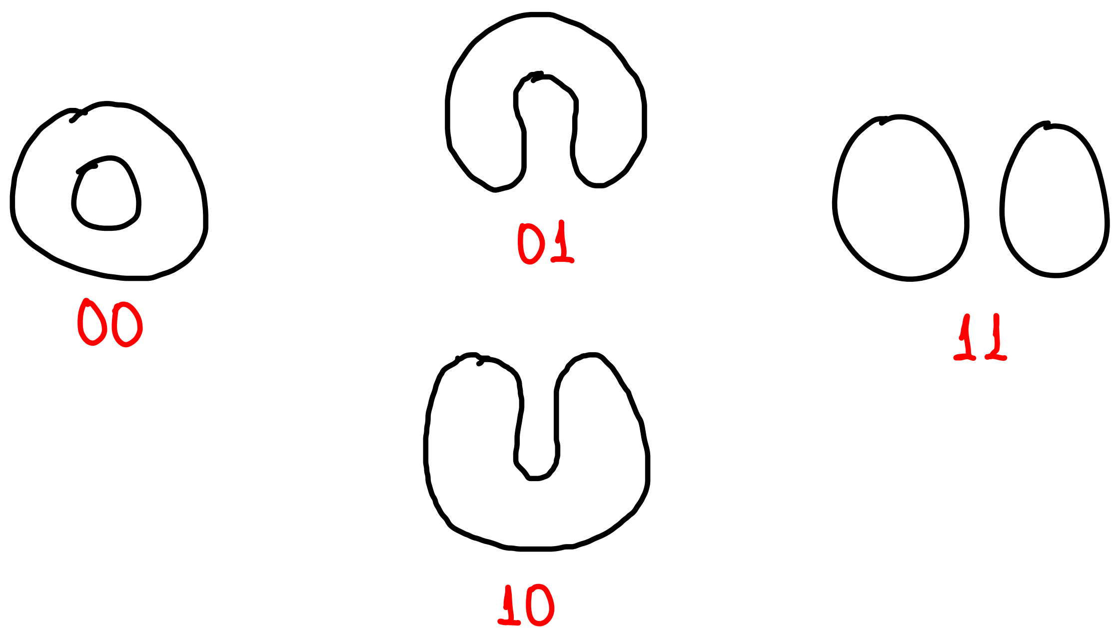


Construction of the Jones polynomial #3

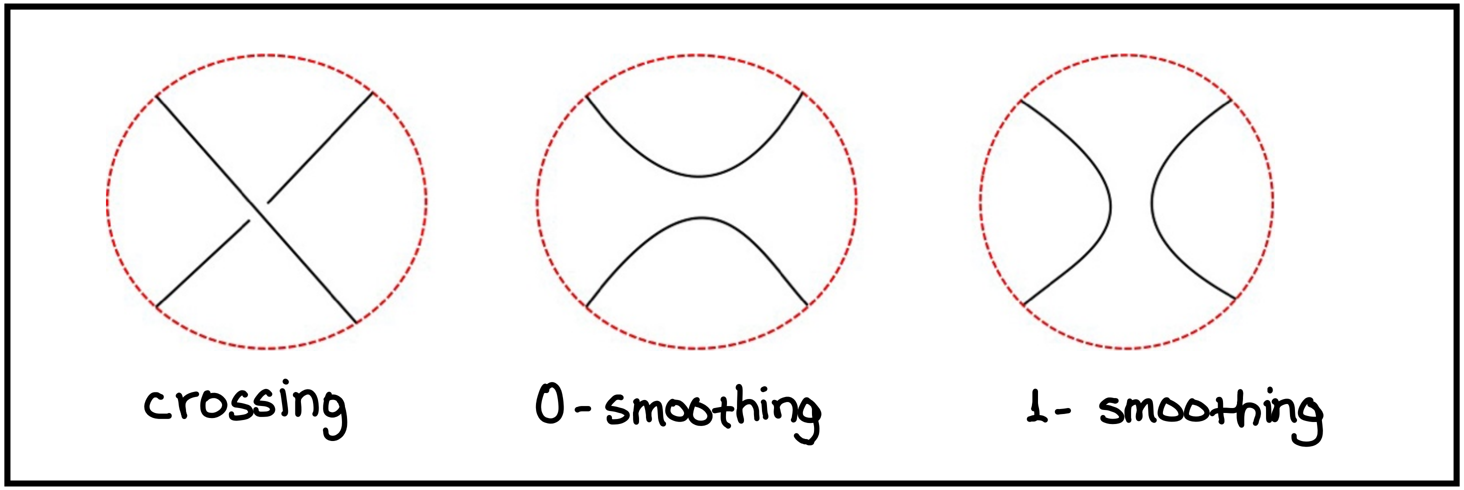


Ex.

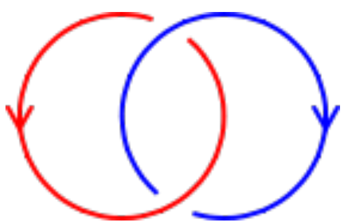


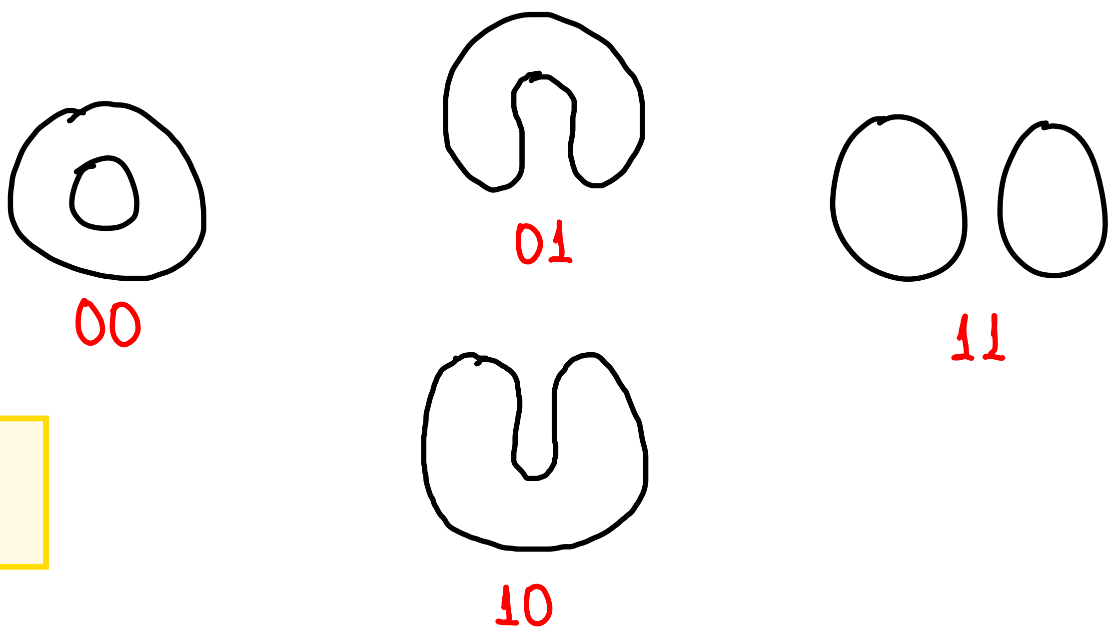


Construction of the Jones polynomial #3



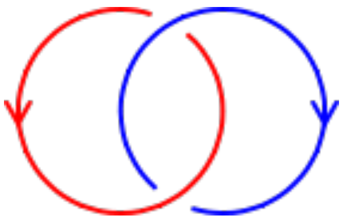
Ex.

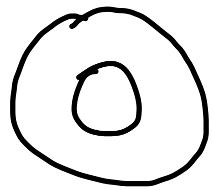




$$(-x)^{\#1\text{'s}}\,(x+x^{-1})^{\#\text{circles}-1}$$

Construction of the Jones polynomial #3





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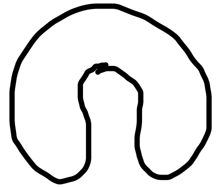

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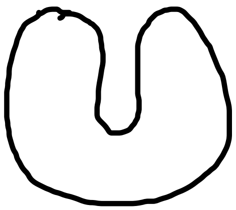

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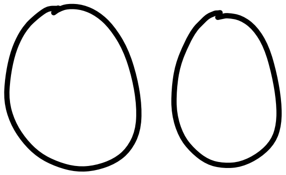

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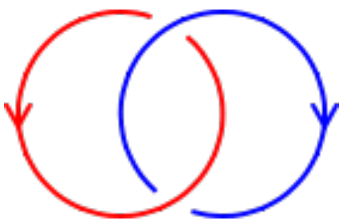
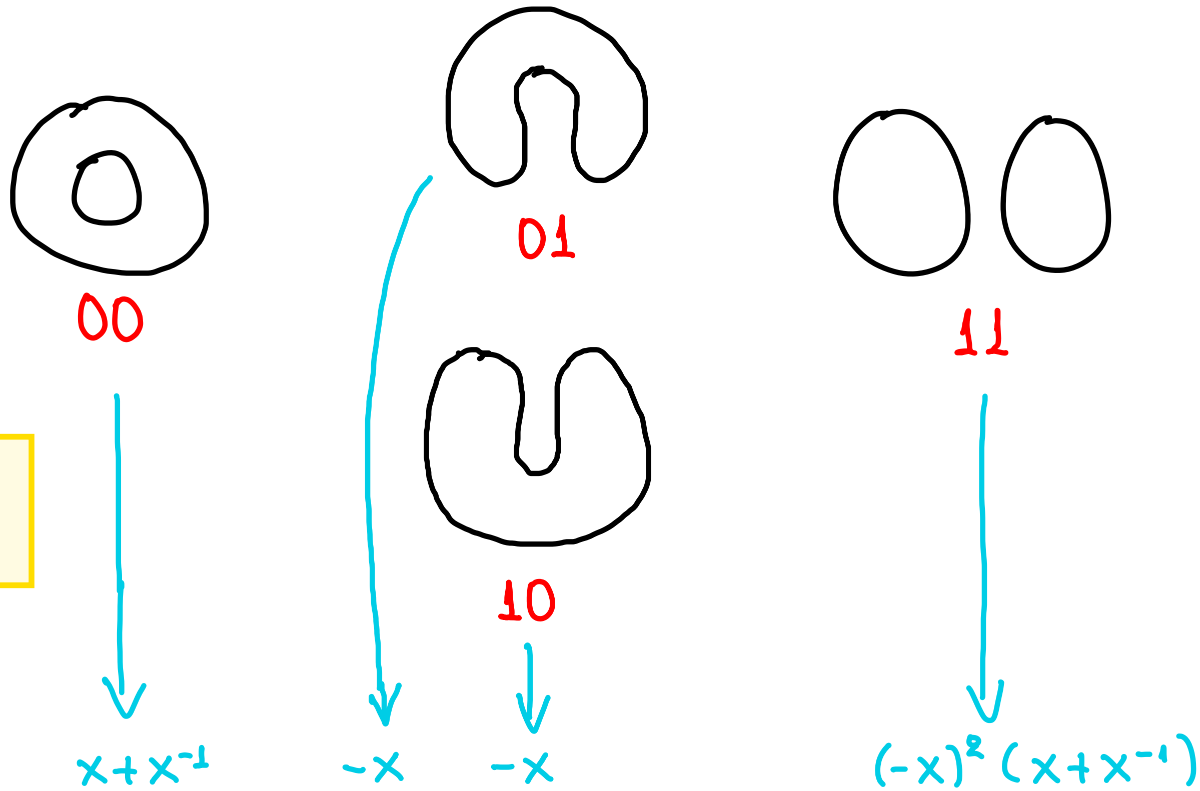
$$\boxed{(-x)^{\#1\text{'s}}\,(x+x^{-1})^{\#\,\text{circles}\,-1}}$$

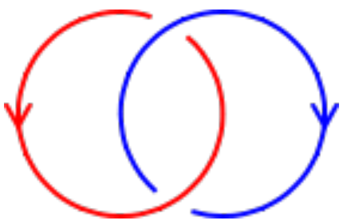
Construction of the Jones polynomial #3

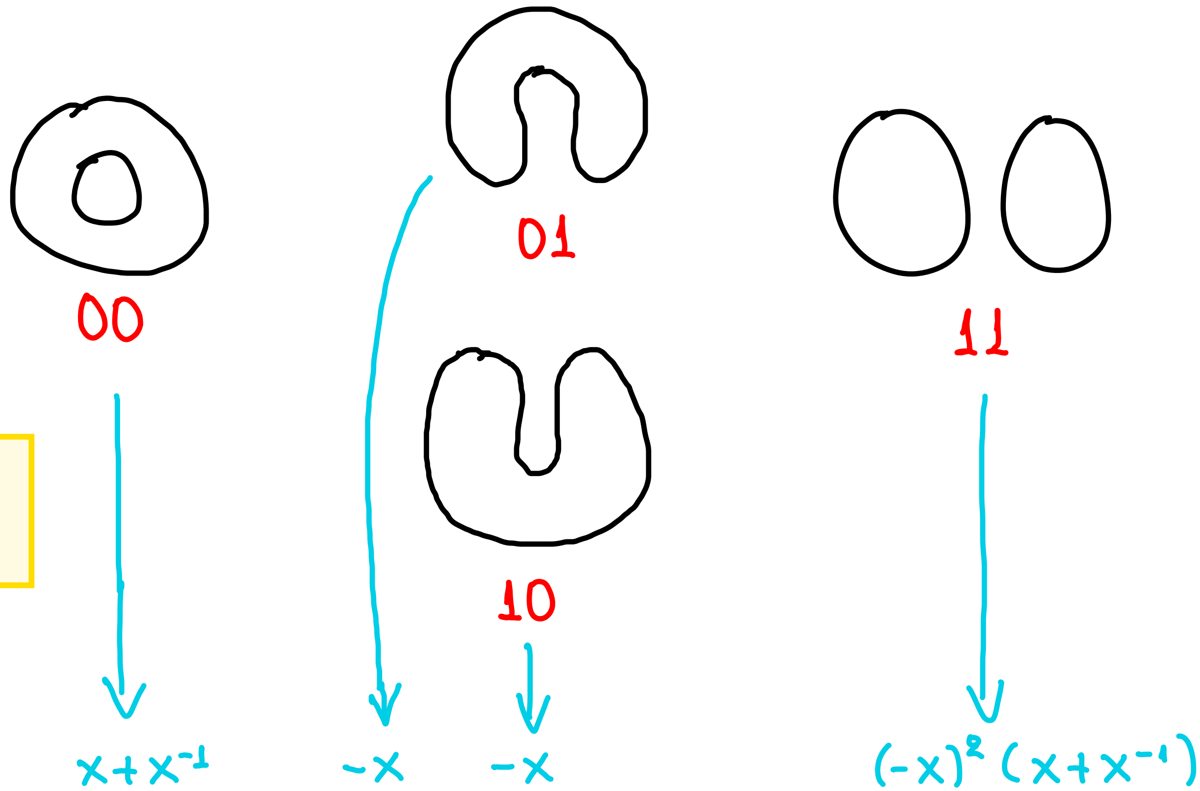


$$\boxed{(-x)^{\#1\text{'s}}\,(x+x^{-1})^{\#\,\text{circles}\,-1}}$$



Construction of the Jones polynomial #3

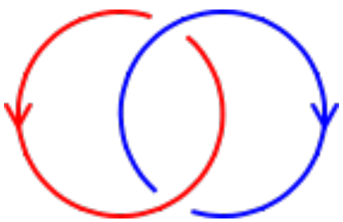




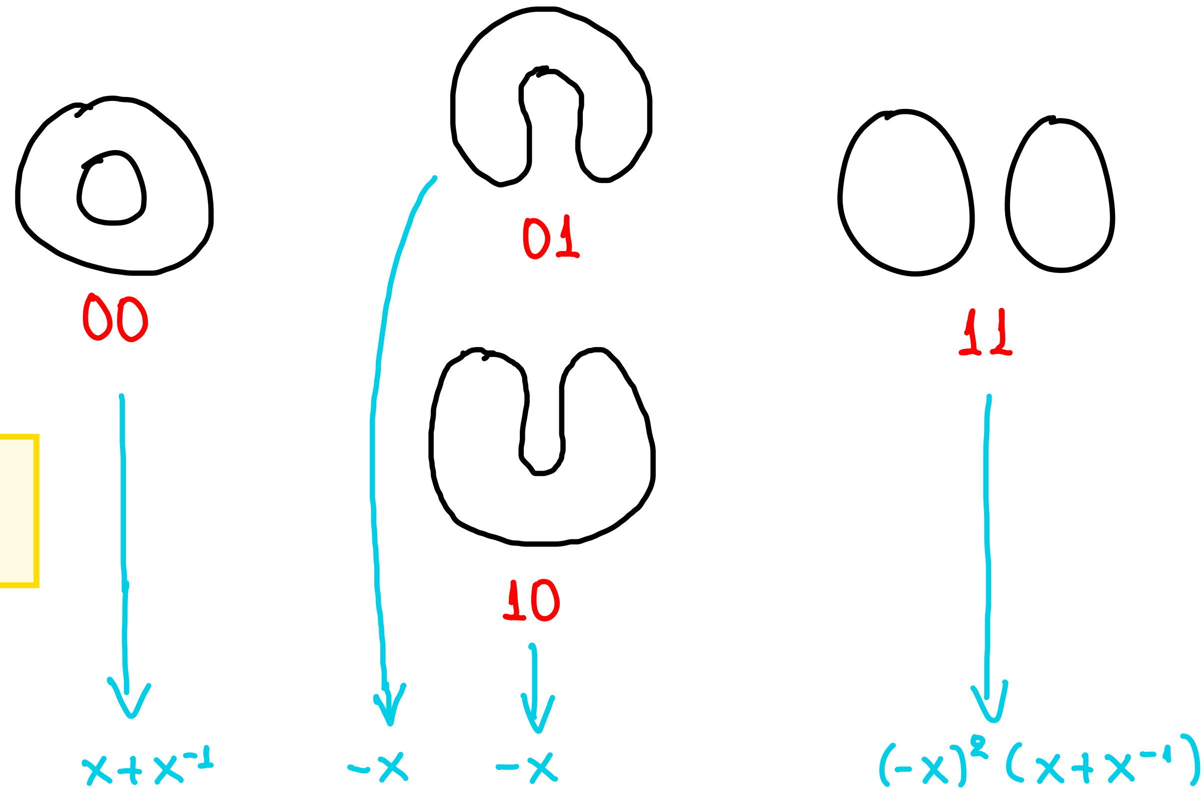
$$\boxed{(-x)^{\#1\text{'s}}(x+x^{-1})^{\#\text{circles}-1}}$$

$$P(\text{right Hopf}) = (x^{-1}+x^{3})$$

Construction of the Jones polynomial #3

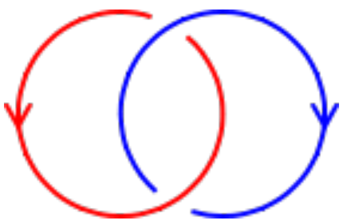


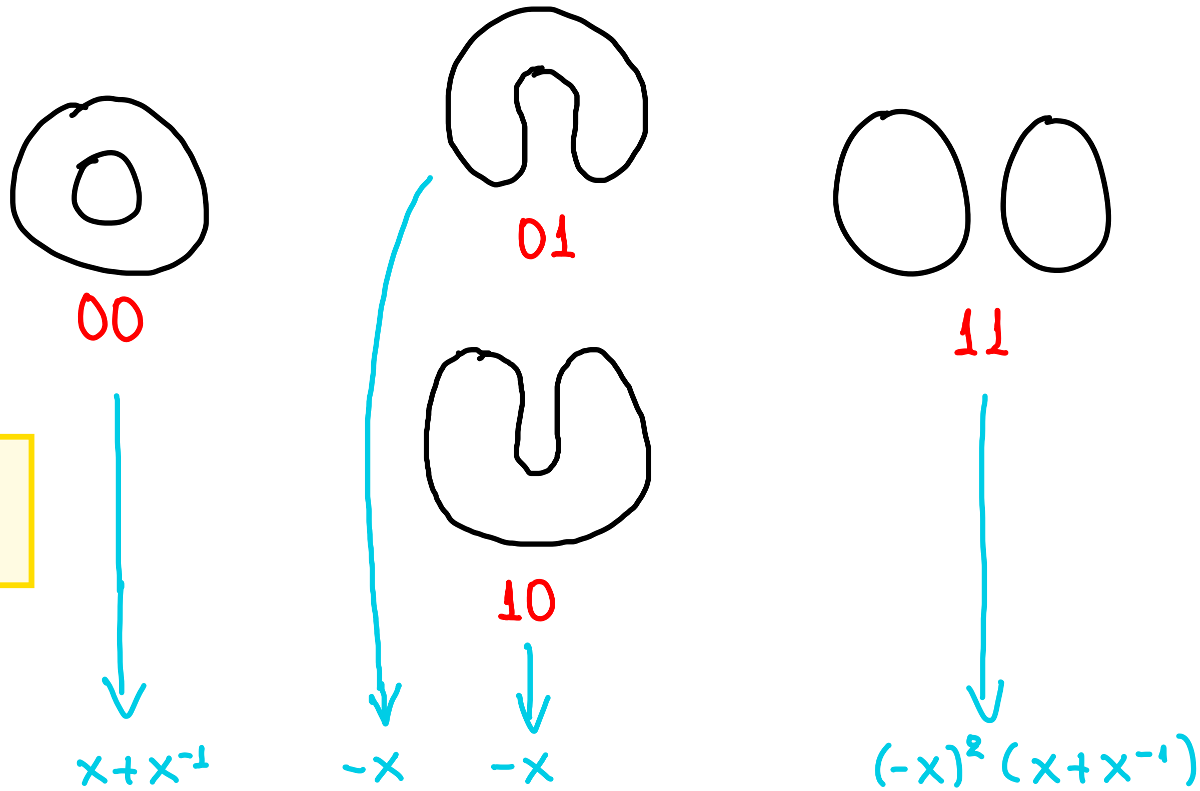
$$\boxed{(-x)^{\#1\text{'s}}\,(x+x^{-1})^{\#\text{circles}-1}}$$



$$P(\text{right Hopf}) = x^2\,(x^{-1}+x^3)$$

Construction of the Jones polynomial #3

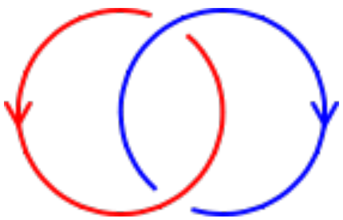




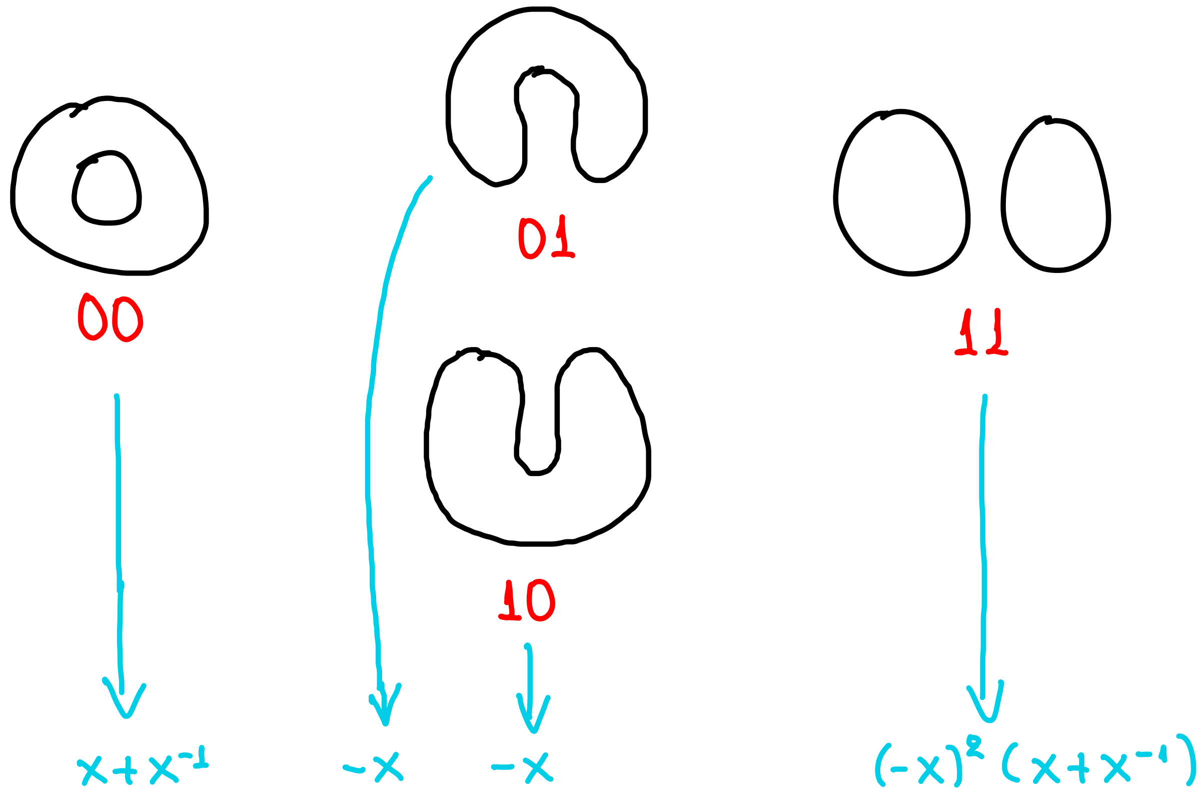
$$\boxed{(-x)^{\#1\text{'s}}\,(x+x^{-1})^{\#\text{circles}-1}}$$

$$P(\text{right Hopf}) = x^2\,(x^{-1}+x^3) = x + x^5$$

Construction of the Jones polynomial #3

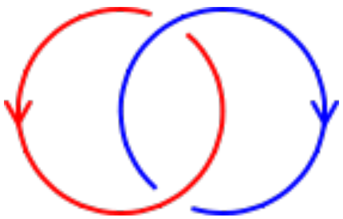


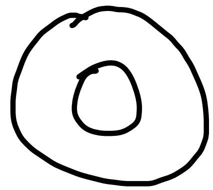
$$\boxed{(-x)^{\#1\text{'s}}\,(x+x^{-1})^{\#\text{circles}}}$$



$$\hat{P}(\text{right Hopf}) = x^2\,(x^{-1}+x^3)(x+x^{-1}) = x + x^5 \cdot (x+x^{-1})$$

Construction of the Jones polynomial #3$^{1/2}$





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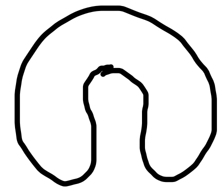

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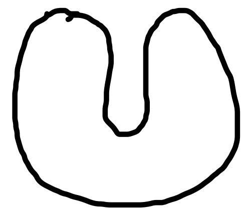

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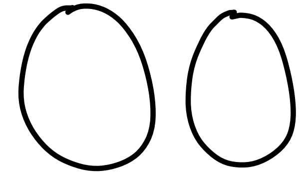

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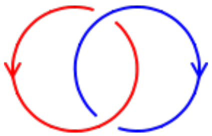
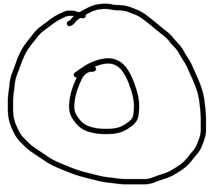
Construction of the Jones polynomial #3½



$V = \langle e_1, e_2 \rangle_{\mathbb{Q}}$



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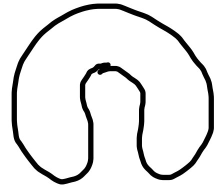

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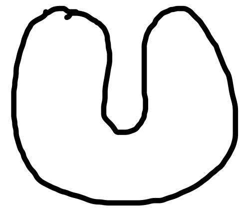

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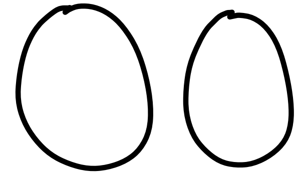

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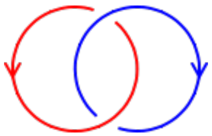
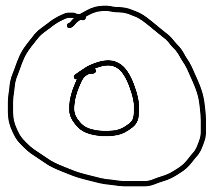
Construction of the Jones polynomial #3½



$$V = \langle e_1, e_2 \rangle_{\mathbb{Q}}$$

$$e_1 \mapsto 1 \qquad e_2 \mapsto -1$$



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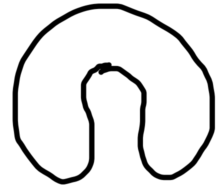

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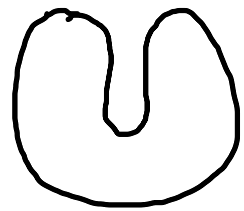

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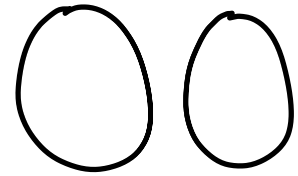

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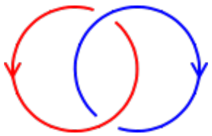
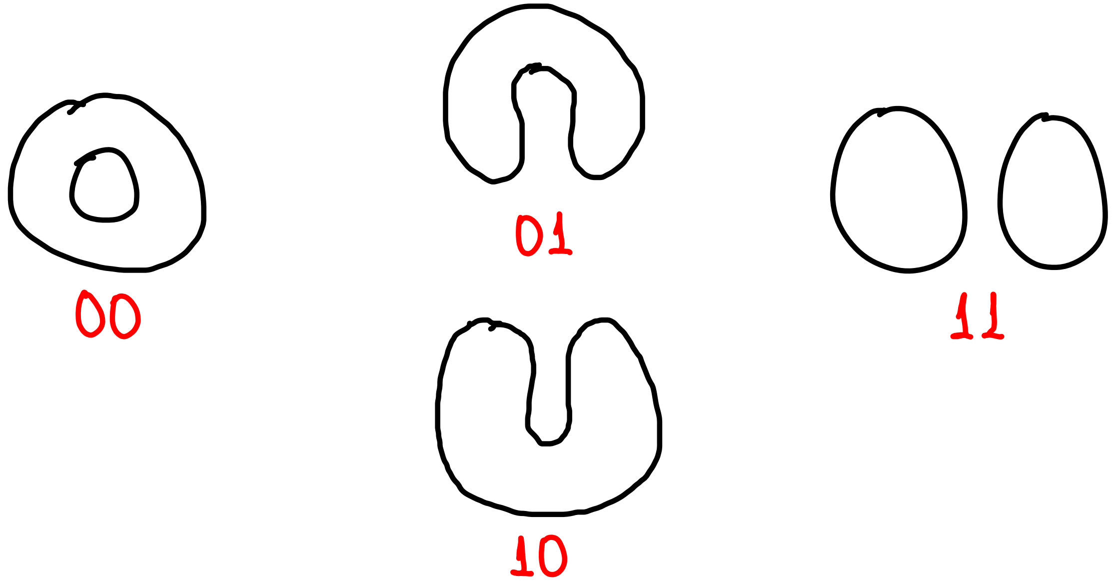
Construction of the Jones polynomial #$3^{1/2}$

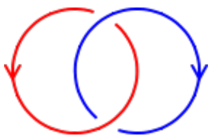


$$V = \langle \underset{\substack{\downarrow\\ 1}}{e_1}, \underset{\substack{\downarrow\\ -1}}{e_2} \rangle_{\mathbb{Q}}$$

$$\dim_x V = x + x^{-1}$$



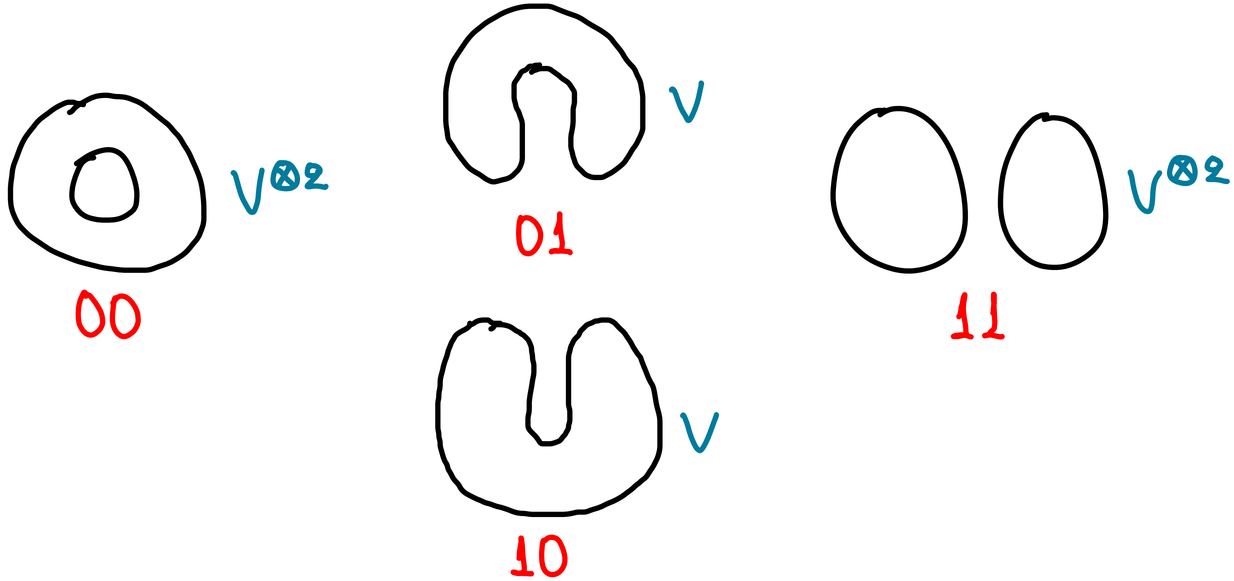
Construction of the Jones polynomial # $3^{1/2}$



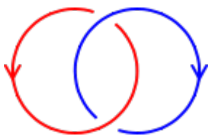
$$V = \langle e_1, e_2 \rangle_{\mathbb{Q}}$$

$e_1 \to 1$, $e_2 \to -1$

$$\dim_x V = x + x^{-1}$$

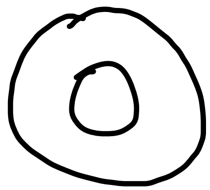


Construction of the Jones polynomial #$3^{1/2}$



$$V = \langle \underset{\underset{1}{\downarrow}}{e_1}, \underset{\underset{-1}{\downarrow}}{e_2} \rangle_{\mathbb{Q}}$$

$$\dim_x V = x + x^{-1}$$



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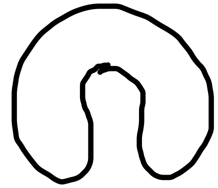
$$C^{0,*} = V^{\otimes 2}$$



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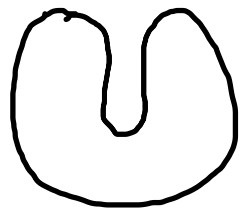

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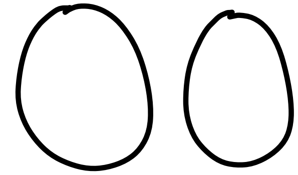
$$C^{1,*} = V \oplus V$$



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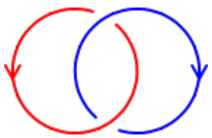
$$C^{2,*} = V^{\otimes 2}$$

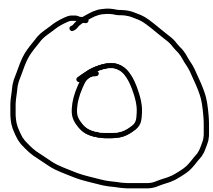
Construction of the Jones polynomial # $3^{1/2}$



$$V = \langle e_1, e_2 \rangle_{\mathbb{Q}}$$

$e_1 \mapsto 1$, $e_2 \mapsto -1$

$$\dim_x V = x + x^{-1}$$



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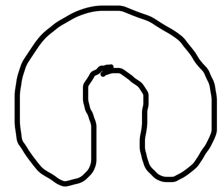
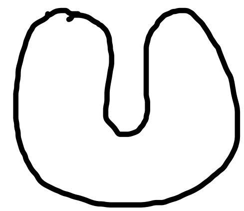
$$C^{0,*} = V^{\otimes 2}\{2\}$$



01



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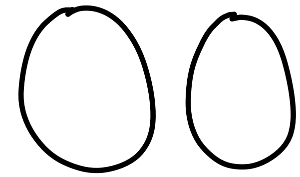
$$C^{1,*} = V\{3\} \oplus V\{3\}$$



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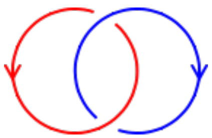
$$C^{2,*} = V^{\otimes 2}\{4\}$$

Construction of the Jones polynomial $\# 3^{1/2}$



$$V = \langle e_1, e_2 \rangle_{\mathbb{Q}}$$

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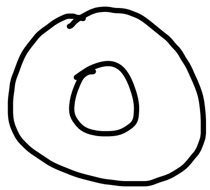

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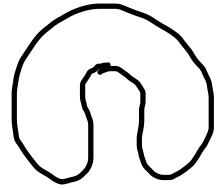

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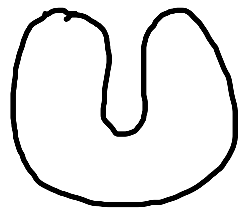

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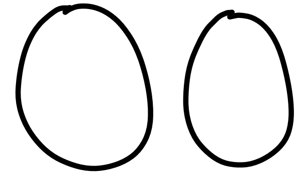

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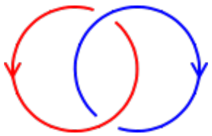
Khovanov homology

$j \backslash i$	0	2
0	$\mathbb{Q}$	
2	$\mathbb{Q}$	
4		$\mathbb{Q}$
6		$\mathbb{Q}$

Construction of the Jones polynomial # $3^{1/2}$



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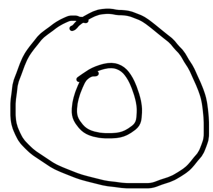

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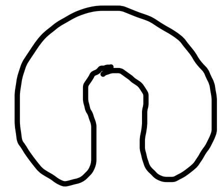

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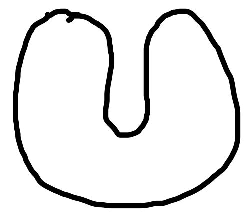

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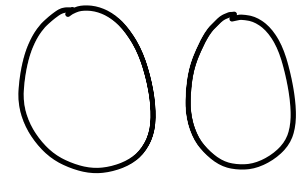

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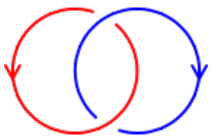

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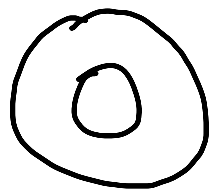

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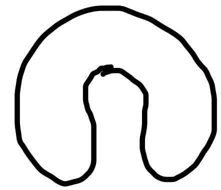

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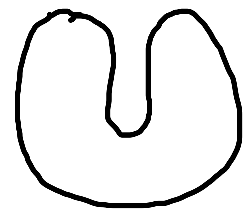

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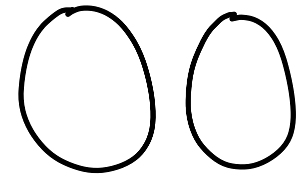

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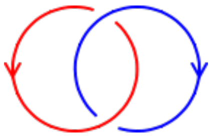

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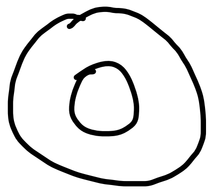

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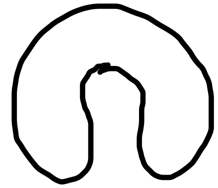

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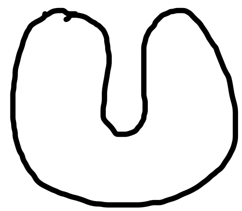

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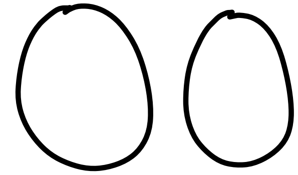

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$$\Big\downarrow /(x + x^{-1})$$

$$P(L) = x + x^5$$

The Yokonuma-Hecke algebra of type A

Let $d \in \mathbb{Z}_{>0}$.

$$Y_{d,n}(q) = \left\langle \begin{array}{c} g_1, \ldots, g_{n-1} \\ t_1, \ldots, t_n \end{array} \middle| \begin{array}{c} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \quad \text{if } |i-j|>1 \\ t_j^d = 1 \quad , t_i t_j = t_j t_i \quad , \quad t_j g_i = g_i t_{s_i(j)} \\ g_i^2 = (q-1) e_i g_i + q \end{array} \right\rangle$$

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$$\mathrm{Irr}(Y_{d,n}(q)) \longleftrightarrow \mathrm{Irr}(\underbrace{\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n}_{\substack{\shortparallel \\ G(d,1,n)}}) \longleftrightarrow \{ d\text{-partitions of } n \} =: \mathcal{P}_d(n)$$

$$\lambda = (\lambda^{(1)}, \ldots, \lambda^{(d)}) \in \mathcal{P}_d(n)$$

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$\tau : Y_{d,n}(q) \longrightarrow \mathbb{C}(q, z, E)$ Markov trace (Juyumaya trace)

$$\begin{aligned} 1 &\longmapsto 1 \\ g_i &\longmapsto z \\ e_i &\longmapsto E \end{aligned}$$

Framed braid group

$$(\mathbb{Z}/d\mathbb{Z}) \wr B_n = \left\langle \sigma_1,\dots,\sigma_{n-1}, t_1,\dots,t_n \;\middle|\; \begin{array}{l} \sigma_i \text{ as before} \\ t_j^d = 1 \quad j=1,\dots,n \\ t_i t_j = t_j t_i \quad i,j=1,\dots,n \\ t_j \sigma_i = \sigma_i t_{s_i(j)} \end{array} \right\rangle$$

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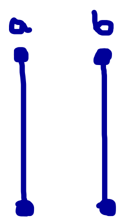
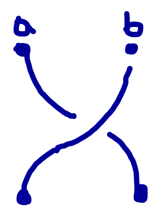
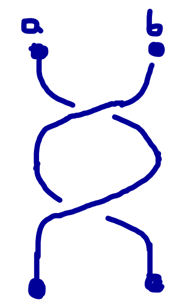
$t_1^a t_2^b =$ 

$t_1^a t_2^b \sigma_1 =$ 

$t_1^a t_2^b \sigma_1^2 =$ 

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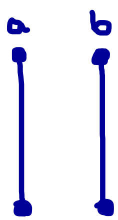
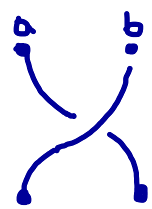
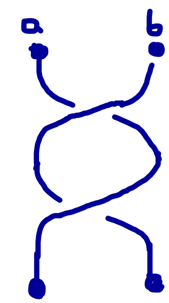
$t_1^a t_2^b =$ 

$t_1^a t_2^b \sigma_1 =$ 

$t_1^a t_2^b \sigma_1^2 =$ 

Multiplication : concatenation of diagrams

<u>Ex.</u> $(t_1^a t_2^b \sigma_1) \cdot (t_1^{a'} t_2^{b'}) = t_1^{a+b'} t_2^{b+a'} \sigma_1$

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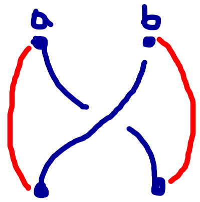
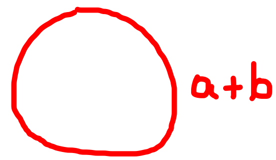
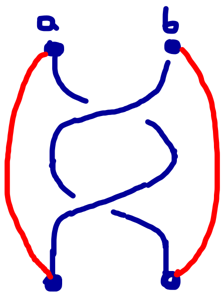
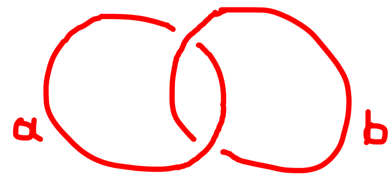
$t_1^a t_2^b \sigma_1 =$ 

$\xrightarrow{\wedge}$ 

$t_1^a t_2^b \sigma_1^2 =$ 

$\xrightarrow{\wedge}$ 

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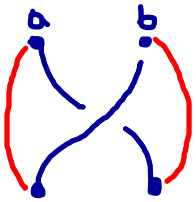
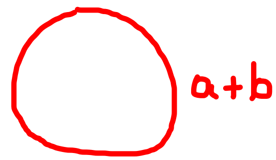
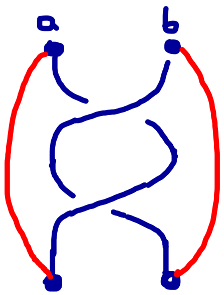
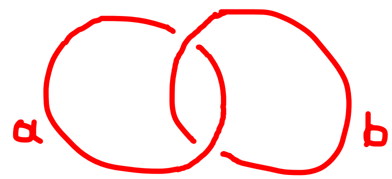
$t_1^a t_2^b \sigma_1 =$  $\xrightarrow{\wedge}$ 

$t_1^a t_2^b \sigma_1^2 =$  $\xrightarrow{\wedge}$ 

$d = 3$: $\quad \widehat{t_1 t_2 \sigma_1} \sim \widehat{t_1^2 \sigma_1} \quad , \quad \widehat{t_1 t_2 \sigma_1^2} \not\sim \widehat{t_1^2 \sigma_1^2}$

The ϑ-invariant : origins

- Construction of an invariant for framed knots and links for $E = \frac{1}{m}$
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The ϑ-invariant : origins

- Construction of an invariant for framed knots and links for $E = \frac{1}{m}$, $m = 1, 2, \ldots, d$ [Juyumaya - Lambropoulou , 2013]

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- Construction of the representations of $Y_{d,n}(q)$ and change of the quadratic relation for the g_i. [C.- Poulain d'Andecy , 2016]

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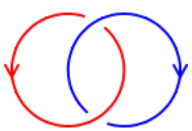  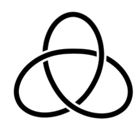 

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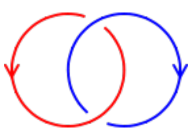

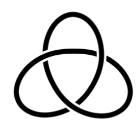

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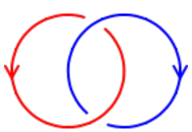   

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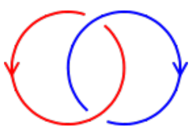  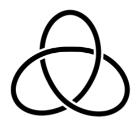 

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- Numerical formula for $\Theta(L)$ using the Homflypt polynomials of all sublinks of L [Appendix by Lickorish]

$\underline{\text{Theorem}}$ [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{d,n}(q) \cong \bigoplus_{\underbrace{\mu_1+\cdots+\mu_d=n}_{\mu \models n}} \mathrm{Mat}_{m_\mu}\left(\mathcal{H}_{\mu_1}(q) \otimes \cdots \otimes \mathcal{H}_{\mu_d}(q)\right) \qquad m_\mu = \frac{n!}{\mu_1! \cdots \mu_d!}$$

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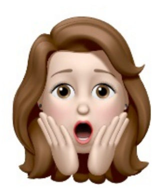
The invariant for classical knots and links arising from $YTL_{d,n}(q)$ is equivalent to the Jones polynomial.



More Temperley-Lieb algebras [GJKL, 2017]

$$FTL_{d,n}(q) := Y_{d,n}(q) \Big/ \Big\langle \sum_{\substack{0 \leq a,b,c < d \\ a+b+c \equiv 0 \bmod d}} t_1^a t_2^b t_3^c G_1 \Big\rangle$$

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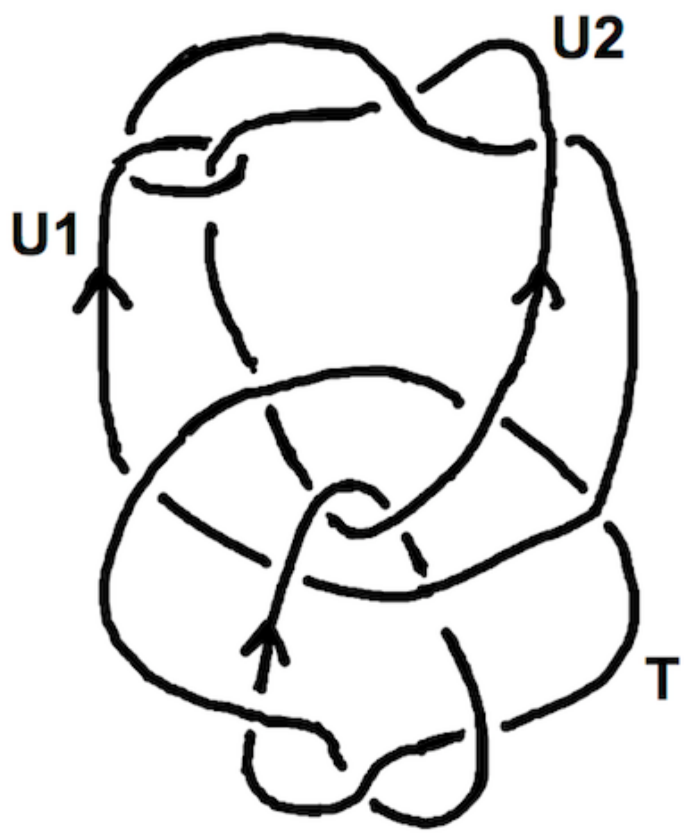
\end{tikzcd}$$

The link LLL(0)

has the same Jones polynomial as the disjoint union of 3 unkots ($1 \in B_3$). However, the θ-invariant distinguishes the two [C., 2019]



A skein relation for the Khovanov homology

The invariant θ can be defined as follows :

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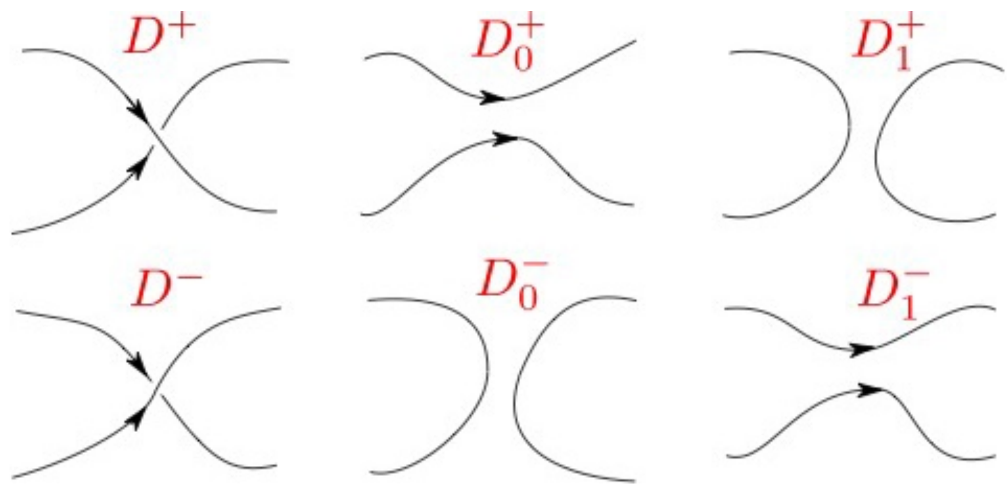

Lemma [C.-Goundaroulis-Kontogeorgis-Lambropoulou, 2021]

$\exists$ exact sequence

$$0 \longrightarrow C^{i-2,j-3}(D_0^+) \longrightarrow C^{i-2,j-4}(D^-) \longrightarrow C^{i,j}(D^+) \longrightarrow C^{i,j-1}(D_0^+) \longrightarrow 0$$

<u>Theorem</u> [C.-Goundaroulis-Kontogeorgis-Lambropoulou, 2021]

The Khovanov polynomial satisfies the generalised skein relation

$$\boxed{t^{-1}x^{-2}\, kh(D^+) - t x^2\, kh(D^-) = (t^{-1}x^{-1} - tx)\, kh(D_0^+) + C(D_0^+, D^-, D^+)}$$

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where

$$C(D_0^+, D^-, D^+) = (t+1)\,x\,Kh(E_2(D_0^{+\,\text{top}})) - (t^{-1}+1)\,x^{-1}\,Kh(E_2(D_0^{+\,\text{bottom}}))$$

where

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<u>Remark:</u> For $t=-1$, $C(D_0^+, D^-, D^+) = 0$ and we get the skein relation of the Jones polynomial.

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Remark: For $t=-1$, $C(D_0^+, D^-, D^+) = 0$ and we get the skein relation of the Jones polynomial.

Example: D^+ = Hopf link
D^- = 2 unknots
D_0^+ = unknot

$$C(D_0^+, D^-, D^+) = 0$$

2.5.2. *The left-handed trefoil knot.* In this example we will study the trefoil knot D^- with three negative crossings. Then D^+ is the unknot and D_0^+ is the Hopf link. The Khovanov homology of the three links is given in the following tables:

$KH(D^-)$:

$j\backslash i$	0	-2	-3
-9			$\mathbb{Q}$
-5		$\mathbb{Q}$	
-3	$\mathbb{Q}$		
-1	$\mathbb{Q}$		

$KH(D^+)$:

$j\backslash i$	0
1	$\mathbb{Q}$
-1	$\mathbb{Q}$

$KH(D_0^+)$:

$j\backslash i$	-2	0
0		$\mathbb{Q}$
-2		$\mathbb{Q}$
-4	$\mathbb{Q}$	
-6	$\mathbb{Q}$	

From the above tables we compute

$$\mathrm{Kh}(D^-) = \frac{1}{t^3q^9} + \frac{1}{t^2q^5} + \frac{1}{q^3} + \frac{1}{q}$$
$$\mathrm{Kh}(D^+) = \frac{1}{q} + q$$
$$\mathrm{Kh}(D_0^+) = \frac{1}{t^2q^6} + \frac{1}{t^2q^4} + \frac{1}{q^2} + 1$$

We thus have:

(i,j)	$KH^{i-2,j-3}(D_0^+)$	$KH^{i-2,j-4}(D^-)$	$KH^{i,j}(D^+)$	$KH^{i,j-1}(D_0^+)$
(-2,-3)				$\mathbb{Q}$
(-2,-5)				$\mathbb{Q}$
(-1,-5)		$\mathbb{Q}$		
(0,1)			$\mathbb{Q}$	$\mathbb{Q}$
(0,-1)	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$
(0,-3)	$\mathbb{Q}$			
(2,3)	$\mathbb{Q}$	$\mathbb{Q}$		
(2,1)	$\mathbb{Q}$	$\mathbb{Q}$		

By looking at the table above, we only need to study the cases where $j \in \{-5, -3, -1, 1, 3\}$.

Case $j = -5$: In the first page, we have the following sequences of Khovanov homology:

$$\begin{tikzcd}
0 \arrow[d] & & & & & 0 \arrow[d] \\
0 = KH^{-4,-8}(D_0^+) \arrow[d] & & & & & 0 = KH^{-3,-8}(D_0^+) \arrow[d] \\
0 = KH^{-4,-9}(D^-) \arrow[d] & & & E_{2,-5}^{-3,-9}(D^-) \arrow[ddl, "\cong"'] & \cong & \mathbb{Q} \cong KH^{-3,-9}(D^-) \arrow[d] \\
0 = KH^{-2,-5}(D^+) \arrow[d] & & & & & 0 = KH^{-1,-5}(D^+) \arrow[d] \\
\mathbb{Q} \cong KH^{-2,-6}(D_0^+) \arrow[d] & \cong & E_{2,-5}^{-2,-6}(D_0^{+b}) & & & 0 = KH^{-1,-6}(D_0^+) \arrow[d] \\
0 & & & & & 0
\end{tikzcd}$$

The above diagonal morphism fits within the third sequence in (2.5) for $i = -1, j = -5$

$$0 \longrightarrow E_{2,-5}^{-3,-9}(D^-) \cong \mathbb{Q} \longrightarrow E_{2,-5}^{-2,-6}(D_0^{+b}) \cong \mathbb{Q} \longrightarrow E_{3,-5}^{-2,-6}(D_0^{+b}) \longrightarrow 0$$

whence $E_{3,-5}^{-2,-6}(D_0^{+b}) = 0$.

Case $j=-3$: In the first page, we have the following sequences of Khovanov homology:

$$\begin{tikzcd}
0 \arrow[d] & & & 0 \arrow[d] \\
0 = KH^{-4,-6}(D_0^+) \arrow[d] & & & 0 = KH^{-3,-6}(D_0^+) \arrow[d] \\
0 = KH^{-4,-7}(D^-) \arrow[d] & & E_{2,-3}^{-3,-7}(D^-) \arrow[r, phantom, "\cong"] \arrow[ddl] & 0 = KH^{-3,-7}(D^-) \arrow[d] \\
0 = KH^{-2,-3}(D^+) \arrow[d] & & & 0 = KH^{-1,-3}(D^+) \arrow[d] \\
\mathbb{Q} \cong KH^{-2,-4}(D_0^+) \arrow[d] \arrow[r, phantom, "\cong"] & E_{2,-3}^{-2,-4}(D_0^{+b}) & & 0 = KH^{-1,-4}(D_0^+) \arrow[d] \\
0 & & & 0
\end{tikzcd}$$

The above diagonal morphism fits within the third sequence in (2.5) for $i=-1, j=-3$

$$0 \longrightarrow E_{2,-3}^{-3,-7}(D^-) \cong 0 \longrightarrow E_{2,-3}^{-2,-4}(D_0^{+b}) \cong \mathbb{Q} \longrightarrow E_{3,-3}^{-2,-4}(D_0^{+b}) \longrightarrow 0$$

whence $E_{3,-3}^{-2,-4}(D_0^{+b}) \cong \mathbb{Q}$.

We also have the following sequences of Khovanov homology:

$$\begin{tikzcd}
0 \arrow[d] & & & 0 \arrow[d] \\
0 = KH^{-3,-6}(D_0^+) \arrow[d] & & E_{2,-3}^{-2,-6}(D_0^{+t}) \arrow[r, phantom, "\cong"] \arrow[ddl] & \mathbb{Q} = KH^{-2,-6}(D_0^+) \arrow[d] \\
0 = KH^{-3,-7}(D^-) \arrow[d] & & & 0 = KH^{-2,-7}(D^-) \arrow[d] \\
0 = KH^{-1,-3}(D^+) \arrow[d] \arrow[r, phantom, "\cong"] & E_{2,-3}^{-1,-3}(D^+) & & 0 = KH^{0,-3}(D^+) \arrow[d] \\
0 = KH^{-1,-4}(D_0^+) \arrow[d] & & & 0 = KH^{0,-4}(D_0^+) \arrow[d] \\
0 & & & 0
\end{tikzcd}$$

The above diagonal morphism fits within the second sequence in (2.5) for $i = 0, j = -3$

$$0 \longrightarrow E_{3,-3}^{-2,-6}(D_0^{+t}) \longrightarrow E_{2,-3}^{-2,-6}(D_0^{+t}) \cong \mathbb{Q} \longrightarrow E_{2,-3}^{-1,-3}(D^+) \cong 0 \longrightarrow 0$$

whence $E_{3,-3}^{-2,-6}(D_0^{+t}) \cong \mathbb{Q}$.

Cases $j \in \{-1, 1, 3\}$: In these cases all vertical sequences are exact so the corresponding elements in the second page are all zero.

$$
\begin{array}{c|c|c}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
KH^{-2,-4}(D_0^+) \cong \mathbb{Q} & KH^{0,-2}(D_0^+) \cong \mathbb{Q} & KH^{0,0}(D_0^+) \cong \mathbb{Q} \\
\downarrow & \downarrow & \downarrow \\
KH^{-2,-5}(D^-) \cong \mathbb{Q} & KH^{0,-3}(D^-) \cong \mathbb{Q} & KH^{0,-1}(D^-) \cong \mathbb{Q} \\
\downarrow & \downarrow & \downarrow \\
KH^{0,-1}(D^+) \cong \mathbb{Q} & KH^{2,1}(D^+) = 0 & KH^{2,3}(D^+) = 0 \\
\downarrow & \downarrow & \downarrow \\
KH^{0,-2}(D_0^+) \cong \mathbb{Q} & KH^{2,0}(D_0^+) = 0 & KH^{2,2}(D_0^+) = 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
$$

There is also a sequence where the Khovanov homology of D^- is equal to 0:

$$
\begin{array}{c}
0 \\
\downarrow \\
KH^{-2,-2}(D_0^+) = 0 \\
\downarrow \\
KH^{-2,-3}(D^-) = 0 \\
\downarrow \\
KH^{0,1}(D^+) \cong \mathbb{Q} \\
\downarrow \\
KH^{0,0}(D_0^+) \cong \mathbb{Q} \\
\downarrow \\
0
\end{array}
$$

We now compute:

$$\mathrm{Kh}(E_2(D_0^{+b})) = \frac{1}{t^2q^6} + \frac{1}{t^2q^4} \quad \text{and} \quad \mathrm{Kh}(E_2(D_0^{+t})) = \frac{1}{t^2q^6}$$

and so

$$
\begin{aligned}
C(D_0^+, D^-, D^+) &= (t+1)q\left(tq^2\mathrm{Kh}(E_2(D_0^{+t})) - \mathrm{Kh}(E_2(D_0^{+b})\right) \\
&= (t+1)q\left(\frac{1}{tq^4} - \frac{1}{t^2q^6} - \frac{1}{t^2q^4}\right) \\
&= (t+1)\left(\frac{tq^2 - 1 - q^2}{t^2q^5}\right).
\end{aligned}
$$

A categorification of the θ-invariant

Idea: - $\mathcal{K}\left(\bigsqcup_{i=1}^{r} K_i\right) = E^{-r}\, Kh\left(\bigsqcup_{i=1}^{r} K_i\right)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}(D^+) - tx^2\,\mathcal{K}(D^-) = (t^{-1}x^{-1} - tx)\,\mathcal{K}(D_0^+) + C(D_0^+, D^-, D^+)$$

A categorification of the θ-invariant

<u>Idea</u>: - $\mathcal{H}(\bigsqcup_{i=1}^{r} K_i) = E^{-r} Kh(\bigsqcup_{i=1}^{r} K_i)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{H}(D^+) - tx^2\,\mathcal{H}(D^-) = (t^{-1}x^{-1} - tx)\,\mathcal{H}(D_0^+) + C(D_0^+, D^-, D^+)$$

<u>Problem</u>: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

A categorification of the θ-invariant

<u>Idea</u>: - $\mathcal{K}'(\bigsqcup_{i=1}^{r} K_i) = E^{-r}\, Kh(\bigsqcup_{i=1}^{r} K_i)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}'(D^+) - tx^2\,\mathcal{K}'(D^-) = (t^{-1}x^{-1} - tx)\underset{\substack{\uparrow \\ E'}}{\,}\mathcal{K}'(D_0^+) + \underset{\substack{\uparrow \\ E^{-\#\text{ components of } D^+}}}{\,} C(D_0^+, D^-, D^+)$$

<u>Problem</u>: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

<u>Solution</u> (?) - Introduce parameters E' and E

A categorification of the θ-invariant

<u>Idea</u>: - $\mathcal{K}'(\coprod_{i=1}^{r} K_i) = E^{-r}\, Kh(\coprod_{i=1}^{r} K_i)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}'(D^+) - t x^2\,\mathcal{K}'(D^-) = \underbrace{(t^{-1}x^{-1} - t x)}_{E'}\,\mathcal{K}'(D_0^+) + \underbrace{}_{E^{-\#\text{ components of } D^+}} C(D_0^+, D^-, D^+)$$

<u>Problem</u>: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

<u>Solution</u> (?) - Introduce parameters E' and E

- Sum over all diagrams with minimal number of crossings

A categorification of the θ-invariant

<u>Idea</u>: - $\mathcal{K}'(\coprod_{i=1}^{r} K_i) = E^{-r}\, Kh(\coprod_{i=1}^{r} K_i)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}'(D^+) - t x^2\,\mathcal{K}'(D^-) = \underbrace{(t^{-1}x^{-1} - tx)}_{E'}\,\mathcal{K}'(D_0^+) + \underbrace{C}_{E^{-\#\text{ components of } D^+}}(D_0^+, D^-, D^+)$$

<u>Problem</u>: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

<u>Solution (?)</u> - Introduce parameters E' and E

- Sum over all diagrams with minimal number of crossings

$\rightsquigarrow$ Categorification of the θ-invariant $\qquad \mathcal{K}'$

A categorification of the θ-invariant

<u>Idea</u>: – $\mathcal{K}'(\bigsqcup_{i=1}^{r} K_i) = E^{-r}\, Kh(\bigsqcup_{i=1}^{r} K_i)$

– Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}'(D^+) - tx^2\,\mathcal{K}'(D^-) = \underset{\substack{\uparrow \\ E'}}{(t^{-1}x^{-1} - tx)}\,\mathcal{K}'(D_0^+) + \underset{\substack{\uparrow \\ E^{-\#\text{ components of } D^+}}}{C(D_0^+, D^-, D^+)}$$

<u>Problem</u>: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

<u>Solution (?)</u> – Introduce parameters E' and E

– Sum over all diagrams with minimal number of crossings

$\rightsquigarrow$ Categorification of the θ-invariant $\qquad \mathcal{K}'_{E,E'}$

A categorification of the ϑ-invariant

<u>Idea</u>: – $\mathcal{K}h'(\bigsqcup_{i=1}^{r} K_i) = E^{-r}\, Kh(\bigsqcup_{i=1}^{r} K_i)$

– Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}h'(D^+) - t x^2\,\mathcal{K}h'(D^-) = \underset{\substack{\uparrow\\ E'}}{(t^{-1}x^{-1} - tx)}\,\mathcal{K}h'(D_0^+) + \underset{\substack{\uparrow\\ E^{-\#\text{ components of } D^+}}}{C(D_0^+, D^-, D^+)}$$

<u>Problem</u>: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

<u>Solution (?)</u> – Introduce parameters E' and E

– Sum over all diagrams with minimal number of crossings

$\rightsquigarrow$ Categorification of the ϑ-invariant

$$\mathcal{K}h'_{E,E'} \xrightarrow[E'=1]{t=-1} \vartheta$$

A categorification of the ϑ-invariant

<u>Idea</u>: - $\mathcal{K}'\left(\bigsqcup_{i=1}^{r} K_i\right) = E^{-r}\, Kh\left(\bigsqcup_{i=1}^{r} K_i\right)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}'(D^+) - t x^2\,\mathcal{K}'(D^-) = (t^{-1}x^{-1} - tx)\underset{\substack{\uparrow\\ E'}}{}\mathcal{K}'(D_0^+) + \underset{\substack{\uparrow\\ E^{-\#\,\text{components of } D^+}}}{} C(D_0^+, D^-, D^+)$$

<u>Problem</u>: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

<u>Solution (?)</u> - Introduce parameters E' and E

- Sum over all diagrams with minimal number of crossings

$\rightsquigarrow$ Categorification of the ϑ-invariant

$$\begin{tikzcd}[column sep=small] & \mathcal{K}'_{E,E'} \arrow[dl, "{t=-1,\ E'=1}"'] \arrow[dr, "{E=E'}"] & \\ \vartheta & & \mathcal{K} \end{tikzcd}$$

A categorification of the θ-invariant

Idea: - $\mathcal{K}'\left(\bigsqcup_{i=1}^{r} K_i\right) = E^{-r}\, Kh\left(\bigsqcup_{i=1}^{r} K_i\right)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}'(D^+) - tx^2\,\mathcal{K}'(D^-) = (t^{-1}x^{-1} - tx)\underset{\substack{\uparrow\\ E'}}{}\mathcal{K}'(D_0^+) + \underset{\substack{\uparrow\\ E^{-\#\text{ components of } D^+}}}{} C(D_0^+, D^-, D^+)$$

Problem: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

Solution (?) - Introduce parameters E' and E

- Sum over all diagrams with minimal number of crossings

$\rightsquigarrow$ Categorification of the θ-invariant

$$\begin{tikzcd} & \mathcal{K}'_{E,E'} \arrow[dl, "{t=-1,\ E'=1}"'] \arrow[dr, "{E=E'}"] & \\ \theta & & \mathcal{K}_E \end{tikzcd}$$

A categorification of the ϑ-invariant

<u>Idea</u>: - $\mathcal{K}'\left(\bigsqcup_{i=1}^{r} K_i\right) = E^{-r}\, Kh\left(\bigsqcup_{i=1}^{r} K_i\right)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}'(D^+) - tx^2\,\mathcal{K}'(D^-) = (t^{-1}x^{-1} - tx)\underset{\substack{\uparrow \\ E'}}{}\mathcal{K}'(D_0^+) + \underset{\substack{\uparrow \\ E^{-\#\text{ components of } D^+}}}{}C(D_0^+, D^-, D^+)$$

<u>Problem</u>: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

<u>Solution (?)</u> - Introduce parameters E' and E
- Sum over all diagrams with minimal number of crossings

$\rightsquigarrow$ Categorification of the ϑ-invariant

$$\begin{tikzcd}
 & \mathcal{K}'_{E,E'} \arrow[dl, "{t=-1,\ E'=1}"'] \arrow[dr, "E=E'"] & \\
\vartheta \arrow[dr, "E=1"] & & \mathcal{K}_E \arrow[dl] \\
 & Kh &
\end{tikzcd}$$

A categorification of the θ-invariant

Idea: - $\mathcal{K}'\left(\bigsqcup_{i=1}^{r} K_i\right) = E^{-r}\, Kh\left(\bigsqcup_{i=1}^{r} K_i\right)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}'(D^+) - t x^2\,\mathcal{K}'(D^-) = (t^{-1}x^{-1} - t x)\underset{\underset{E'}{\uparrow}}{}\mathcal{K}'(D_0^+) + \underset{\underset{E^{-\#\text{ components of } D^+}}{\uparrow}}{} C(D_0^+, D^-, D^+)$$

Problem: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

Solution (?) - Introduce parameters E' and E

- Sum over all diagrams with minimal number of crossings

$\rightsquigarrow$ Categorification of the θ-invariant

$$\begin{tikzcd}
 & \mathcal{K}'_{E,E'} \arrow[dl, "\substack{t=-1\\ E'=1}"'] \arrow[dr, "E=E'"] & \\
\theta \arrow[dr] & & \mathcal{K}_E \arrow[dl] \\
 & \overset{E=1}{Kh} &
\end{tikzcd}$$

A categorification of the ϑ-invariant

Idea: - $\mathcal{K}'\left(\bigsqcup_{i=1}^{r} K_i\right) = E^{-r}\, Kh\left(\bigsqcup_{i=1}^{r} K_i\right)$

- Apply the generalised skein relation to mixed crossings

$$t^{-1}x^{-2}\,\mathcal{K}'(D^+) - t x^2\,\mathcal{K}'(D^-) = (t^{-1}x^{-1} - tx)\underset{\underset{E'}{\uparrow}}{}\mathcal{K}'(D_0^+) + \underset{\underset{E^{-\#\,\text{components of } D^+}}{\uparrow}}{}C(D_0^+, D^-, D^+)$$

Problem: $C(D_0^+, D^-, D^+)$ depends on all 3 diagrams

Solution (?) - Introduce parameters E' and E

- Sum over all diagrams with minimal number of crossings

$\rightsquigarrow$ Categorification of the ϑ-invariant

$$\begin{tikzcd}
 & \mathcal{K}'_{E,E'} \arrow[dl, "{t=-1,\ E'=1}"'] \arrow[dr, "E=E'"] & \\
\vartheta \arrow[dr] & & \mathcal{K}_E \arrow[dl] \\
 & \overset{E=1}{Kh} &
\end{tikzcd}$$