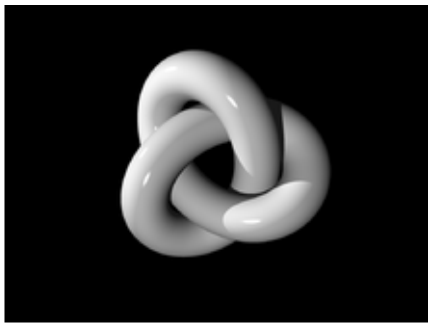
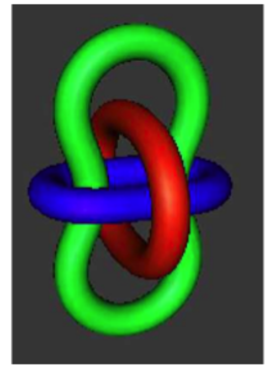
THE YOKONUMA-HECKE ALGEBRA OF TYPE A

Maria Chlouveraki (UVSQ)

for the 50 years of Chevalley Seminar

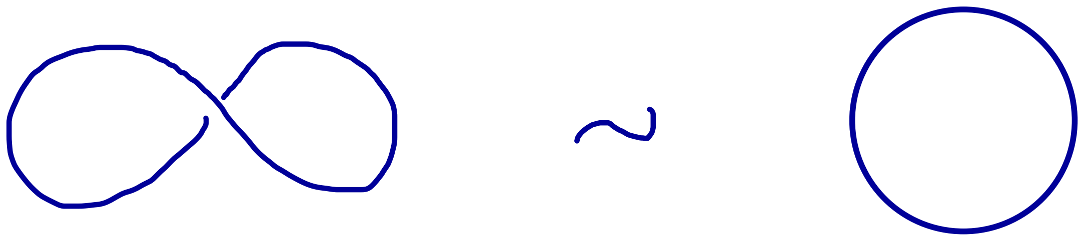
A **knot** (respectively a **link**) is an embedding of the circle S^1 (resp. n copies of S^1) into 3-dimensional Euclidean space $\mathbb{R}^3$.



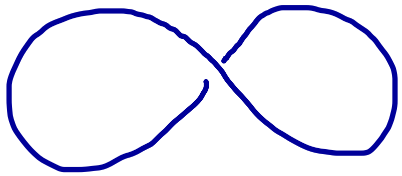


Every link can be represented by a "knot diagram"

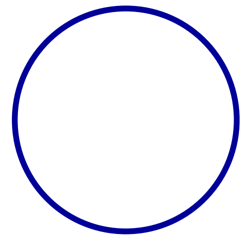
Two knots (or links) are **equivalent** if there is an ambient isotopy between them



Two knots (or links) are equivalent if there is an ambient isotopy between them

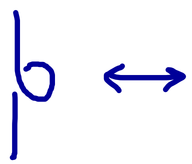


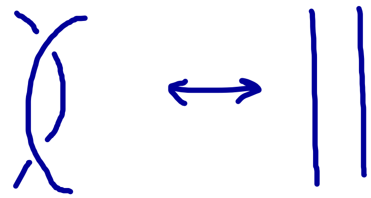
$\sim$

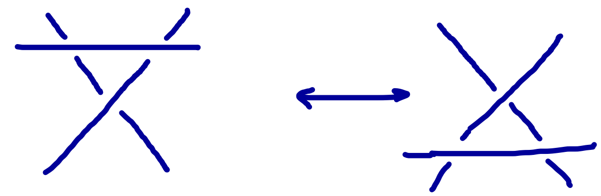


NO!

Theorem (Reidemeister 1935) : Two knots are equivalent if and only if their knot diagrams differ by a finite number of the following moves :

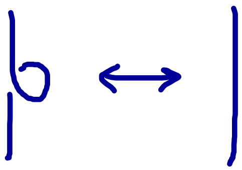
I. 

II. 

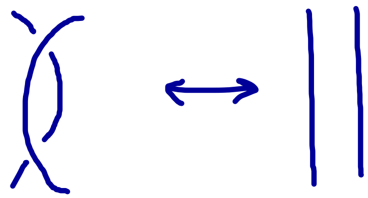
III. 

Theorem (Reidemeister 1935): Two knots are equivalent if and only if their knot diagrams differ by a finite number of the following moves:

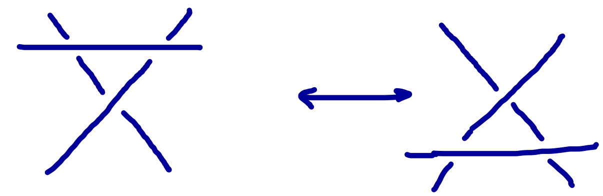
I.



II.



III.



$\mathcal{L}$ = set of links

A knot invariant is a function $I : \mathcal{L} \longrightarrow S$ (S a set)

such that

$$L_1 \sim L_2 \implies I(L_1) = I(L_2)$$

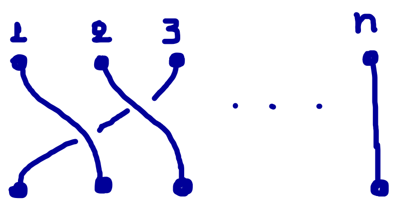
for $L_1, L_2 \in \mathcal{L}$.

Braid group (of type A)

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i = 1, \ldots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| > 1 \end{array} \right\rangle$$

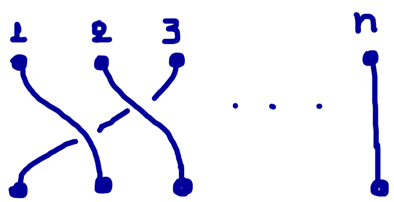
Braid group (of type A)

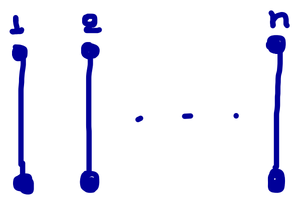
$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i = 1, \ldots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| > 1 \end{array} \right\rangle$$

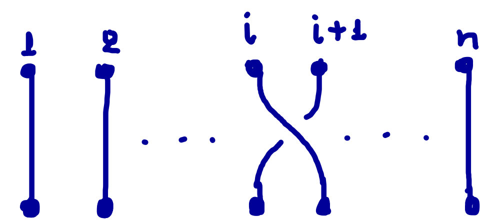


Braid group (of type A)

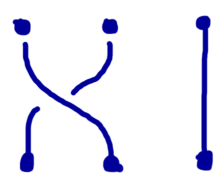
$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i = 1, \ldots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| > 1 \end{array} \right\rangle$$

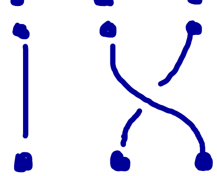


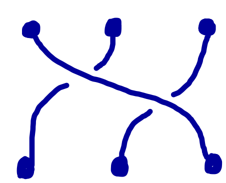
Identity = 

σ_i = 

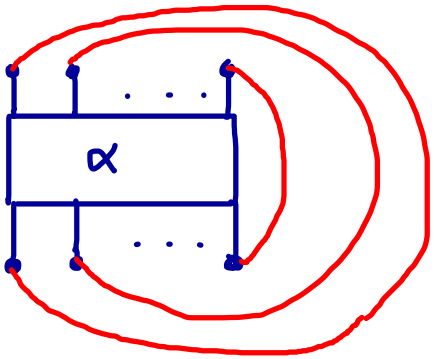
Multiplication : concatenation of diagrams

E.g. $\alpha = \sigma_1 =$ 

$\beta = \sigma_2 =$ 

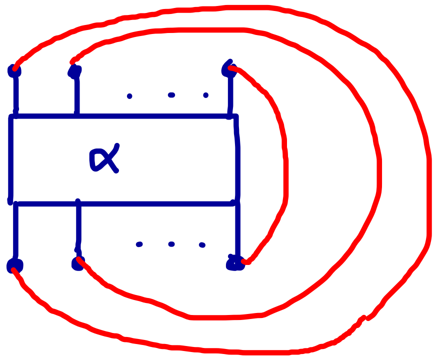
$\Rightarrow \alpha\beta = \sigma_1 \sigma_2 =$ 

Every element of B_n gives rise to a knot or link

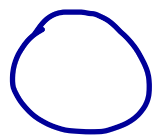


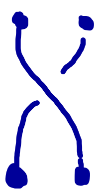
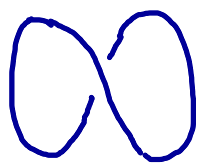
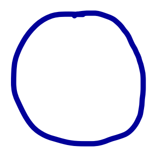
$=: \hat{\alpha} =$ closure of α

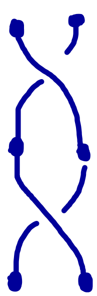
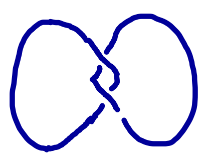
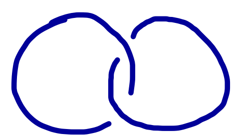
Every element of B_n gives rise to a knot or link



$=: \hat{\alpha}$ = closure of α

<u>E.g.</u> $\alpha =$  $\hat{\alpha} =$ 

$\alpha = \sigma_1 =$  $\hat{\alpha} =$  $\sim$ 

$\alpha = \sigma_1^2 =$  $\hat{\alpha} =$  $\sim$ 

$\alpha = \sigma_1^3$ $\hat{\alpha}$ = left trefoil knot

Alexander's Theorem (1923)

Every link can be obtained as the closure $\hat{\alpha}$ of a braid $\alpha \in \bigcup_{n \geq 1} B_n$.

Alexander's Theorem (1923)

Every link can be obtained as the closure $\hat{\alpha}$ of a braid $\alpha \in \bigcup_{n \geq 1} B_n$.

We define an equivalence relation on $\bigcup_{n \geq 1} B_n$ as the transitive closure of the relations:

(i) $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in B_n$ (conjugation)

(ii) $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n$ (Markov's move)

Alexander's Theorem (1923)

Every link can be obtained as the closure $\hat{\alpha}$ of a braid $\alpha \in \bigcup_{n\geq 1} B_n$.

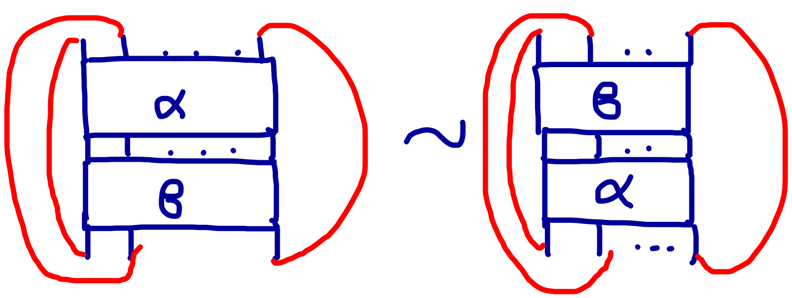
We define an equivalence relation on $\bigcup_{n\geq 1} B_n$ as the transitive closure of the relations:

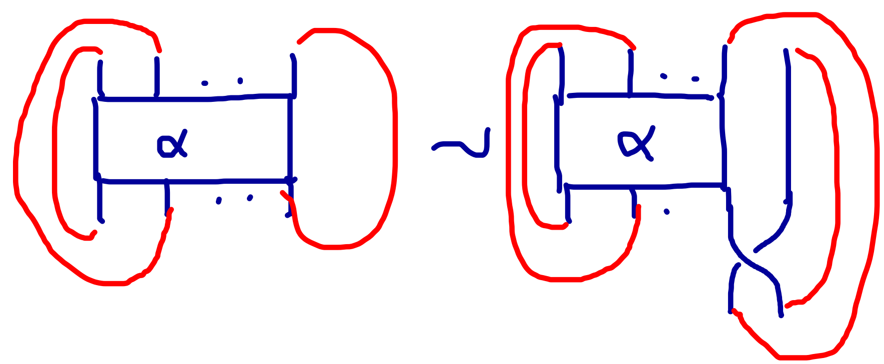
(i) $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in B_n$ (conjugation)

(ii) $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n$ (Markov's move)

Markov's Theorem (1935)

We have $\hat{\alpha} \sim \hat{\beta}$ if and only if $\alpha \sim \beta$.





Iwahori - Hecke algebra of type A

q indeterminate , $R = \mathbb{C}[q, q^{-1}]$

$$\mathcal{H}_n(q) = \left\langle G_1, \dots, G_{n-1} \;\middle|\; \begin{array}{l} G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} \\ G_i G_j = G_j G_i \quad \text{if } |i-j| > 1 \\ G_i^2 = (q-1) G_i + q \end{array} \right\rangle$$

Iwahori - Hecke algebra of type A

q indeterminate , $R = \mathbb{C}[q, q^{-1}]$

$$\mathcal{H}_n(q) = \left\langle G_1, \dots, G_{n-1} \;\middle|\; \begin{array}{l} G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} \\ G_i G_j = G_j G_i \quad \text{if } |i-j| > 1 \\ G_i^2 = (q-1) G_i + q \end{array} \right\rangle$$

- $\mathcal{H}_n(q)$ is a quotient of $R[B_n]$

Iwahori - Hecke algebra of type A

q indeterminate , $R = \mathbb{C}[q, q^{-1}]$

$$\mathcal{H}_n(q) = \left\langle G_1, \dots, G_{n-1} \;\middle|\; \begin{array}{l} G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} \\ G_i G_j = G_j G_i \quad \text{if } |i-j| > 1 \\ G_i^2 = (q-1) G_i + q \end{array} \right\rangle$$

- $\mathcal{H}_n(q)$ is a quotient of $R[B_n]$
- $q = 1$: $\mathcal{H}_n(1) \cong \mathbb{C}[S_n]$

Iwahori - Hecke algebra of type A

q indeterminate, $R = \mathbb{C}[q, q^{-1}]$

$$\mathcal{H}_n(q) = \left\langle G_1, \dots, G_{n-1} \;\middle|\; \begin{array}{l} G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} \\ G_i G_j = G_j G_i \quad \text{if } |i-j| > 1 \\ G_i^2 = (q-1) G_i + q \end{array} \right\rangle$$

- $\mathcal{H}_n(q)$ is a quotient of $R[B_n]$
- $q = 1$: $\mathcal{H}_n(1) \cong \mathbb{C}[S_n]$
- $\mathcal{H}_n(q)$ is a free R-module of rank $n! = |S_n|$
  $(B = \{ G_w = G_{i_1} \dots G_{i_r} \mid w = s_{i_1} \dots s_{i_r} \in S_n \})$

reduced expression, $s_i = (i, i+1)$

G_{n-1} appears at most once

Iwahori - Hecke algebra of type A

q indeterminate, $R = \mathbb{C}[q, q^{-1}]$

$$\mathcal{H}_n(q) = \left\langle G_1, \dots, G_{n-1} \;\middle|\; \begin{array}{l} G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} \\ G_i G_j = G_j G_i \quad \text{if } |i-j| > 1 \\ G_i^2 = (q-1) G_i + q \end{array} \right\rangle$$

- $\mathcal{H}_n(q)$ is a quotient of $R[B_n]$
- $q = 1$: $\mathcal{H}_n(1) \cong \mathbb{C}[S_n]$
- $\mathcal{H}_n(q)$ is a free R-module of rank $n! = |S_n|$
  $\left(B = \{ G_w = G_{i_1} \dots G_{i_r} \mid w = s_{i_1} \dots s_{i_r} \in S_n \} \right)$

reduced expression , $s_i = (i, i+1)$

G_{n-1} appears at most once

$$\boxed{R = \mathcal{H}_1(q) \subset \mathcal{H}_2(q) \subset \dots \subset \mathcal{H}_n(q) \subset \mathcal{H}_{n+1}(q) \subset \dots}$$

Theorem (Jones - Ocneanu 1987)

Let $z \in \mathbb{C}$. There exists a unique linear map $\tau : \mathcal{H}_n(q) \longrightarrow R$ such that

- $\tau(1) = 1$
- $\tau(ab) = \tau(ba) \quad \forall\ a, b \in \mathcal{H}_n(q)$
- $\tau(\alpha G_{k-1}) = z\,\tau(\alpha) \quad \forall\ k = 1, \dots, n$ and $\alpha \in \mathcal{H}_{k-1}(q)$

Theorem (Jones - Ocneanu 1987)

Let $z \in \mathbb{C}$. There exists a unique linear map $\tau : \mathcal{H}_n(q) \longrightarrow R$ such that

- $\tau(1) = 1$
- $\tau(ab) = \tau(ba) \quad \forall\ a, b \in \mathcal{H}_n(q)$
- $\tau(\alpha G_{k-1}) = z\,\tau(\alpha) \quad \forall\ k = 1, \ldots, n$ and $\alpha \in \mathcal{H}_{k-1}(q)$

$$\boxed{B_n \hookrightarrow R[B_n] \twoheadrightarrow \mathcal{H}_n(q) \xrightarrow{\ \tau\ } R}$$

Theorem (Jones - Ocneanu 1987)

Let $z \in \mathbb{C}$. There exists a unique linear map $\tau : \mathcal{H}_n(q) \longrightarrow R$ such that

- $\tau(1) = 1$
- $\tau(ab) = \tau(ba) \quad \forall\, a,b \in \mathcal{H}_n(q)$
- $\tau(\alpha G_{k-1}) = z\,\tau(\alpha) \quad \forall\, k = 1, \dots, n$ and $\alpha \in \mathcal{H}_{k-1}(q)$

$$\boxed{B_n \hookrightarrow R[B_n] \twoheadrightarrow \mathcal{H}_n(q) \xrightarrow{\tau} R}$$

$\underline{z \neq 0:}$

$\alpha, \beta \in B_n$: $P(\alpha) = c(\alpha) \cdot \tau(\alpha)$ such that $c(\alpha\beta) = c(\beta\alpha)$

$P(\alpha) = P(\alpha\sigma_n) = P(\alpha\sigma_n^{-1})$

HOMFLYPT or 2-variable Jones polynomial

Framed braid group

Let $d \in \mathbb{Z}_{>0}$

$$(\mathbb{Z}/d\mathbb{Z}) \wr B_n = \left\langle \sigma_1,\dots,\sigma_{n-1}, t_1,\dots,t_n \;\middle|\; \begin{array}{ll} \sigma_i \text{ as before} & \\ t_j^d = 1 & j=1,\dots,n \\ t_i t_j = t_j t_i & i,j=1,\dots,n \\ t_j \sigma_i = \sigma_i t_{s_i(j)} & \end{array} \right\rangle$$

$$\parallel$$

$$(\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$$

where $s_i = (i, i+1) \in S_n$

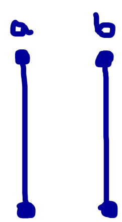
Framed braid group

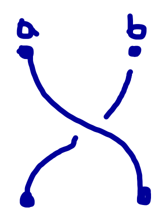
Let $d \in \mathbb{Z}_{>0}$

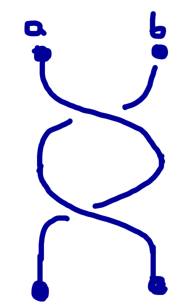
$$
\begin{array}{c}(\mathbb{Z}/d\mathbb{Z}) \wr B_n \\ \parallel \\ (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n\end{array} = \left\langle \sigma_1,\dots,\sigma_{n-1}, t_1,\dots,t_n \;\middle|\; \begin{array}{ll} \sigma_i \text{ as before} & \\ t_j^d = 1 & j=1,\dots,n \\ t_i t_j = t_j t_i & i,j=1,\dots,n \\ t_j \sigma_i = \sigma_i t_{s_i(j)} & \end{array} \right\rangle
$$

where $s_i = (i, i+1) \in S_n$

E.g. $a, b \in \{0,1,\dots,d-1\}$, $n=2$

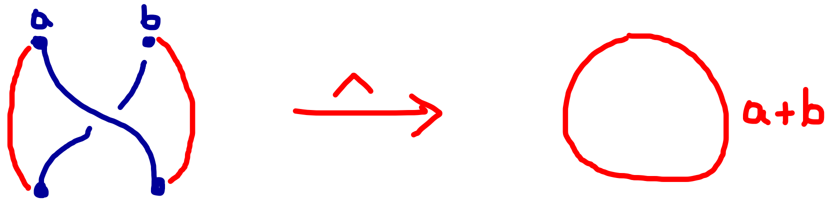
$t_1^a t_2^b =$ 

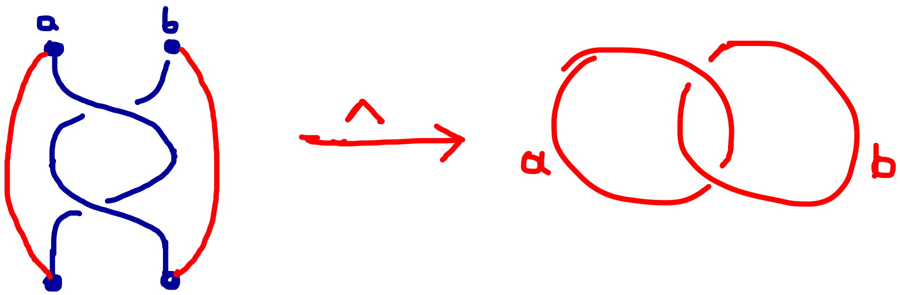
$t_1^a t_2^b \sigma_1 =$ 

$t_1^a t_2^b \sigma_1^2 =$ 

Every element of $(\mathbb{Z}/d\mathbb{Z}) \wr B_n$ gives rise to a framed knot or link

<u>E.g.</u> $a, b \in \{0, 1, \dots, d-1\}$, $n = 2$

$t_1^a t_2^b \sigma_1 =$ 

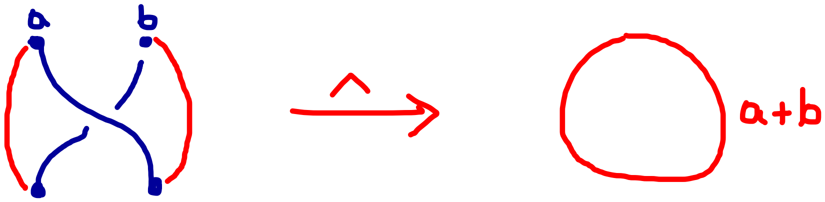
$t_1^a t_2^b \sigma_1^2 =$ 

$d = 3$: $\quad \widehat{t_1 t_2 \sigma_1} \sim \widehat{t_1^2 \sigma_1}$, $\quad \widehat{t_1 t_2 \sigma_1^2} \not\sim \widehat{t_1^2 \sigma_1^2}$

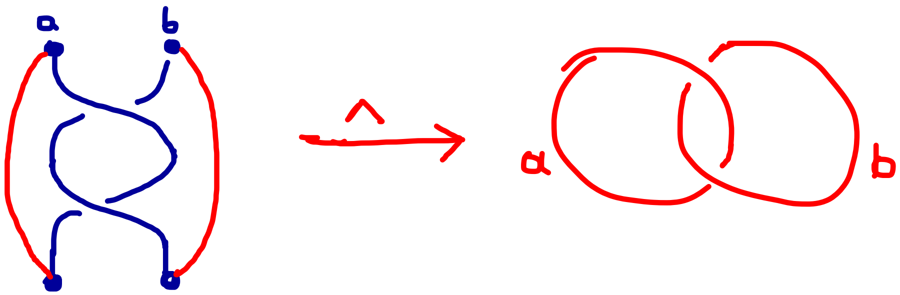
Every element of $(\mathbb{Z}/d\mathbb{Z}) \wr B_n$ gives rise to a framed knot or link.

E.g. $a, b \in \{0, 1, \dots, d-1\}$, $n = 2$

$t_1^a t_2^b \sigma_1 =$



$t_1^a t_2^b \sigma_1^2 =$



$d = 3$: $\widehat{t_1 t_2 \sigma_1} \sim \widehat{t_1^2 \sigma_1}$, $\widehat{t_1 t_2 \sigma_1^2} \not\sim \widehat{t_1^2 \sigma_1^2}$

Alexander's Theorem : obvious

Markov's Theorem : Ko-Smolinsky 1992

Yokonuma - Hecke algebra of type A

q indeterminate , $R = \mathbb{C}[q, q^{-1}]$

$$Y_{d,n}(q) = \left\langle \begin{array}{c} g_1, \ldots, g_{n-1} \\ t_1, \ldots, t_n \end{array} \;\middle|\; \begin{array}{l} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \quad \text{if} \quad |i-j| > 1 \\ t_j^d = 1 \quad , \; t_i t_j = t_j t_i \quad , \quad t_j g_i = g_i t_{s_i(j)} \\ \qquad g_i^2 = 1 + (q-1) e_i + (q-1) e_i g_i \end{array} \right\rangle$$

where $\quad e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}$ is an idempotent.

Yokonuma-Hecke algebra of type A

q indeterminate, $R = \mathbb{C}[q, q^{-1}]$

$$Y_{d,n}(q) = \left\langle \begin{matrix} g_1, \dots, g_{n-1} \\ t_1, \dots, t_n \end{matrix} \;\middle|\; \begin{matrix} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \quad \text{if } |i-j|>1 \\ t_j^d = 1 \quad , \; t_i t_j = t_j t_i \quad , \quad t_j g_i = g_i t_{s_i(j)} \\ g_i^2 = 1 + (q-1) e_i + (q-1) e_i g_i \end{matrix} \right\rangle$$

where $\displaystyle e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}$ is an idempotent.

- $Y_{d,n}(q)$ is a quotient of $R[(\mathbb{Z}/d\mathbb{Z}) \wr B_n]$

Yokonuma-Hecke algebra of type A

q indeterminate, $R = \mathbb{C}[q, q^{-1}]$

$$Y_{d,n}(q) = \left\langle \begin{matrix} g_1, \dots, g_{n-1} \\ t_1, \dots, t_n \end{matrix} \;\middle|\; \begin{matrix} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \quad \text{if} \quad |i-j| > 1 \\ t_j^d = 1 \quad , t_i t_j = t_j t_i \quad , \quad t_j g_i = g_i t_{s_i(j)} \\ g_i^2 = 1 + (q-1) e_i + (q-1) e_i g_i \end{matrix} \right\rangle$$

where $e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}$ is an idempotent.

- $Y_{d,n}(q)$ is a quotient of $R[(\mathbb{Z}/d\mathbb{Z}) \wr B_n]$
- $q = 1$: $Y_{d,n}(1) \cong \mathbb{C}[G(d,1,n)]$, where $G(d,1,n) = (\mathbb{Z}/d\mathbb{Z}) \wr S_n$

Yokonuma-Hecke algebra of type A

q indeterminate, $R = \mathbb{C}[q, q^{-1}]$

$$Y_{d,n}(q) = \left\langle \begin{matrix} g_1, \dots, g_{n-1} \\ t_1, \dots, t_n \end{matrix} \;\middle|\; \begin{matrix} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \quad \text{if } |i-j|>1 \\ t_j^d = 1 \quad , \quad t_i t_j = t_j t_i \quad , \quad t_j g_i = g_i t_{s_i(j)} \\ g_i^2 = 1 + (q-1) e_i + (q-1) e_i g_i \end{matrix} \right\rangle$$

where $\displaystyle e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}$ is an idempotent.

- $Y_{d,n}(q)$ is a quotient of $R[(\mathbb{Z}/d\mathbb{Z}) \wr B_n]$
- $q = 1$: $Y_{d,n}(1) \cong \mathbb{C}[G(d,1,n)]$, where $G(d,1,n) = (\mathbb{Z}/d\mathbb{Z}) \wr S_n$
- $d = 1$: $Y_{1,n}(q) \cong \mathcal{H}_n(q)$

Yokonuma - Hecke algebra of type A

q indeterminate, $R = \mathbb{C}[q,q^{-1}]$

$$Y_{d,n}(q) = \left\langle \begin{matrix} g_1,\dots,g_{n-1} \\ t_1,\dots,t_n \end{matrix} \;\middle|\; \begin{matrix} g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}, \quad g_ig_j = g_jg_i \quad \text{if } |i-j|>1 \\ t_j^d = 1 \quad , \; t_it_j = t_jt_i \quad , \quad t_jg_i = g_it_{s_i(j)} \\ g_i^2 = 1 + (q-1)e_i + (q-1)e_ig_i \end{matrix} \right\rangle$$

where $\quad e_i := \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}$ is an idempotent.

- $Y_{d,n}(q)$ is a quotient of $R[(\mathbb{Z}/d\mathbb{Z})\wr B_n]$
- $q = 1$: $Y_{d,n}(1) \cong \mathbb{C}[G(d,1,n)]$, where $G(d,1,n) = (\mathbb{Z}/d\mathbb{Z})\wr S_n$
- $d = 1$: $Y_{1,n}(q) \cong \mathcal{H}_n(q)$
- $\mathcal{H}_n(q)$ is a quotient of $Y_{d,n}(q)$ $\quad (t_j \longmapsto 1)$

- $Y_{d,n}(q)$ is a free R-module of rank $d^n \cdot n! = |G(d,1,n)|$

  ($\mathcal{B} = \{ t_1^{r_1} \dots t_n^{r_n} g_w \mid r_1, \dots, r_n \in \{0, 1, \dots d-1\}, w \in S_n \}$

  [Juyumaya 2004])

- $Y_{d,n}(q)$ is a free R-module of rank $d^n \cdot n! = |G(d,1,n)|$

  ($\mathcal{B} = \{ t_1^{r_1} \dots t_n^{r_n} g_w \mid r_1, \dots, r_n \in \{0,1,\dots d-1\}, w \in S_n \}$)

  [Juyumaya 2004]

$$\boxed{R = Y_{d,0}(q) \subset Y_{d,1}(q) \subset \cdots \subset Y_{d,n}(q) \subset Y_{d,n+1}(q) \subset \cdots}$$

- $Y_{d,n}(q)$ is a free R-module of rank $d^n \cdot n! = |G(d,1,n)|$
  $\left(\begin{array}{l} B = \{ t_1^{r_1} \cdots t_n^{r_n} g_w \mid r_1, \ldots, r_n \in \{0,1,\ldots d-1\}, \ w \in S_n \} \\ \text{[Juyumaya 2004]} \end{array} \right)$

$$\boxed{R = Y_{d,0}(q) \subset Y_{d,1}(q) \subset \cdots \subset Y_{d,n}(q) \subset Y_{d,n+1}(q) \subset \cdots}$$

<u>Theorem</u> (Juyumaya 2004)

Let $z \in \mathbb{C}$ and $(x_0, x_1, \ldots, x_{d-1}) \in \mathbb{C}^d$ with $x_0 = 1$.

There exists a unique linear map $tr : Y_{d,n}(q) \longrightarrow R$ such that

- $tr(1) = 1$
- $tr(ab) = tr(ba) \quad \forall \ a, b \in Y_{d,n}(q)$
- $tr(a g_{k-1}) = z \cdot tr(a) \quad \forall \ k = 1, \ldots, n$ and $a \in Y_{d,k-1}(q)$
- $tr(a t_k^m) = x_m tr(a)$ —— " —— , $m \in \{0,1,\ldots,d-1\}$

$$\boxed{(\mathbb{Z}/d\mathbb{Z}) \wr B_n \hookrightarrow R[(\mathbb{Z}/d\mathbb{Z}) \wr B_n] \twoheadrightarrow Y_{d,n}(q) \xrightarrow{tr} R}$$

$$\boxed{(\mathbb{Z}/d\mathbb{Z}) \wr B_n \hookrightarrow R[(\mathbb{Z}/d\mathbb{Z}) \wr B_n] \twoheadrightarrow Y_{d,n}(q) \xrightarrow{tr} R}$$

<u>$z \neq 0$:</u>

$$\text{Normalisation of } tr \;\Big\downarrow\; \text{E-condition on } (x_0, x_1, \ldots, x_{d-1})$$

Juyumaya-Lambropoulou invariant for framed knots

$$\boxed{(\mathbb{Z}/d\mathbb{Z}) \wr B_n \hookrightarrow R[(\mathbb{Z}/d\mathbb{Z}) \wr B_n] \twoheadrightarrow Y_{d,n}(q) \xrightarrow{tr} R}$$

$z \neq 0$:

Normalisation of tr

$\downarrow$ E-condition on $(x_0, x_1, \ldots, x_{d-1})$

Juyumaya-Lambropoulou invariant for framed knots

$\downarrow$ "Forget" the framings

Invariant for classical knots and links

$$\boxed{(\mathbb{Z}/d\mathbb{Z}) \wr B_n \hookrightarrow R[(\mathbb{Z}/d\mathbb{Z}) \wr B_n] \twoheadrightarrow Y_{d,n}(q) \xrightarrow{tr} R}$$

<u>$z \neq 0$:</u>

$$\begin{array}{c} \text{Normalisation of } tr \\ \Big\downarrow \quad \text{E-condition on } (x_0, x_1, \ldots, x_{d-1}) \\ \text{Juyumaya-Lambropoulou} \\ \text{invariant for framed knots} \\ \Big\downarrow \quad \text{"Forget" the framings} \\ \text{Invariant for classical knots and links} \underset{\text{C.-L.}}{=} \text{HOMFLYPT when } tr(e_i)=1 \end{array}$$

Representation theory of $Y_{d,n}(q)$

- Thiem 2005 : Unipotent Hecke algebras

- C.-Poulain d'Andecy : Jucys-Murphy elements
  Explicit combinatorial formulas
  Semisimplicity criterion / Schur elements

Representation theory of $Y_{d,n}(q)$

- Thiem 2005 : Unipotent Hecke algebras
- C.-Poulain d'Andecy : Jucys-Murphy elements
  Explicit combinatorial formulas
  Semisimplicity criterion / Schur elements

$$\boxed{\mathrm{Irr}(\mathbb{C}(q)Y_{d,n}(q)) \longleftrightarrow \mathrm{Irr}(G(d,1,n)) \longleftrightarrow \{\text{d-partitions of } n\}}$$

A d-partition of n is a family of d partitions $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)})$ such that $|\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(d)}| = n$.

$\mathbb{C}[G(d,1,n)]$

<u>Yokonuma - Hecke algebra</u>

- Wreath product
- Braid group of type A
- Rep^n theory of $\mathcal{H}_n(q)$

<u>Ariki - Koike algebra</u>

- Quadratic relation for g_i
- Braid group of type B
- $\mathcal{H}_n(q)$ (obvious) subalgebra

$$\mathbb{C}[G(d,1,n)]$$

<u>Yokonuma - Hecke algebra</u>

- Wreath product
- Braid group of type A
- Rep^n theory of $\mathcal{H}_n(q)$

<u>Ariki - Koike algebra</u>

- Quadratic relation for g_i
- Braid group of type B
- $\mathcal{H}_n(q)$ (obvious) subalgebra

<u>Q</u>. What is the connection between $Y_{d,n}(q)$ and $\mathcal{H}_n(q)$?

Cyclotomic Yokonuma-Hecke algebra $Y(d,m,n)$

[C. - Poulain d'Andecy]

Yokonuma-Hecke algebra

$Y_{d,n}(q) = Y(d,1,n)$

Ariki-Koike algebra

$Y(1,m,n)$

Cyclotomic Yokonuma-Hecke algebra $Y(d,m,n)$

[C. - Poulain d'Andecy]

Yokonuma-Hecke algebra

$Y_{d,n}(q) = Y(d,1,n)$

Ariki-Koike algebra

$Y(1,m,n)$

- Markov trace on the Ariki-Koike algebra [Lambropoulou, Geck-Lambropoulou 1994-1999]

$\downarrow$

Invariant for knots in the solid torus

Cyclotomic Yokonuma-Hecke algebra $Y(d,m,n)$

[C. - Poulain d'Andecy]

Yokonuma-Hecke algebra

$Y_{d,n}(q) = Y(d,1,n)$

Ariki-Koike algebra

$Y(1,m,n)$

• Markov trace on the Ariki-Koike algebra [Lambropoulou, Geck-Lambropoulou 1994-1999]

$\downarrow$

Invariant for knots in the solid torus

• Markov trace on $Y(d,m,n)$

$\downarrow$ E-condition

Invariant for framed knots in the solid torus

$\downarrow$ Forget the framings

Invariant for knots in the solid torus

Temperley-Lieb algebra

Let $n \geqslant 3$.

$$TL_n(q) := \mathcal{H}_n(q) \Big/ \langle 1 + G_1 + G_2 + G_1G_2 + G_2G_1 + G_1G_2G_1 \rangle$$

Temperley-Lieb algebra

Let $n \geqslant 3$.

$$TL_n(q) := \mathcal{H}_n(q) \big/ \langle 1 + G_1 + G_2 + G_1G_2 + G_2G_1 + G_1G_2G_1 \rangle$$

$$\operatorname{Irr}(\mathbb{C}(q)\mathcal{H}_n(q)) \longleftrightarrow \{\text{partitions of } n\}$$

$$\operatorname{Irr}(\mathbb{C}(q)\,TL_n(q)) \longleftrightarrow \left\{ \begin{array}{c} \text{partitions of } n \text{ whose Young diagram} \\ \text{has at most 2 columns} \end{array} \right\}$$

Temperley-Lieb algebra

Let $n \geqslant 3$.

$$TL_n(q) := \mathcal{H}_n(q) \big/ \langle 1 + G_1 + G_2 + G_1G_2 + G_2G_1 + G_1G_2G_1 \rangle$$

$$\mathrm{Irr}\,(\mathbb{C}(q)\mathcal{H}_n(q)) \longleftrightarrow \{ \text{partitions of } n \}$$

$$\mathrm{Irr}\,(\mathbb{C}(q)\,TL_n(q)) \longleftrightarrow \left\{ \begin{array}{c} \text{partitions of } n \text{ whose Young diagram} \\ \text{has at most 2 columns} \end{array} \right\}$$

Basis $\nearrow$ Easily extracted from the standard basis of $\mathcal{H}_n(q)$

Basis $\searrow$ Diagrammatic

Yokonuma-Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$YTL_{d,n}(q) := Y_{d,n}(q) \big/ \langle 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1 \rangle$$

Yokonuma-Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$YTL_{d,n}(q) := Y_{d,n}(q) \big/ \langle 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1 \rangle$$

[C.-Pouchin]

$$\mathrm{Irr}\left(\mathbb{C}(q) YTL_{d,n}(q)\right) \longleftrightarrow \left\{ \begin{array}{l} d\text{-partitions } \lambda = (\lambda^{(1)}, \dots, \lambda^{(d)}) \text{ of } n \\ \text{such that the Young diagrams of} \\ \text{all } \lambda^{(i)} \text{ together have at most 2 columns} \end{array} \right\}$$

Yokonuma-Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$YTL_{d,n}(q) := Y_{d,n}(q) \,/\, \langle 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1 \rangle$$

[C.-Pouchin]

$$\mathrm{Irr}\left(\mathbb{C}(q) YTL_{d,n}(q)\right) \longleftrightarrow \left\{ \begin{array}{l} d\text{-partitions } \lambda = (\lambda^{(1)}, \dots, \lambda^{(d)}) \text{ of } n \\ \text{such that the Young diagrams of} \\ \text{all } \lambda^{(i)} \text{ together have at most 2 columns} \end{array} \right\}$$

Basis : extracted from the standard basis of $Y_{d,n}(q)$ (with difficulty!)

Framisation of the Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$FTL_{d,n}(q) := Y_{d,n}(q) \,/\, \langle e_1 e_2 \,(1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1) \rangle$$

Framisation of the Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$FTL_{d,n}(q) := Y_{d,n}(q) \,/ \left\langle e_1 e_2 \left(1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1 \right) \right\rangle$$

[C.-Pouchin]

$$\mathrm{Irr}\left(\mathbb{C}(q)\, FTL_{d,n}(q)\right) \longleftrightarrow \left\{ \begin{array}{l} d\text{-partitions } \lambda = (\lambda^{(1)}, \ldots, \lambda^{(d)}) \text{ of } n \\ \text{such that the Young diagram of} \\ \text{each } \lambda^{(i)} \text{ has at most 2 columns} \end{array} \right\}$$

Framisation of the Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou]

Let $n \geqslant 3$.

$$FTL_{d,n}(q) := Y_{d,n}(q) \big/ \langle e_1 e_2 (1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1) \rangle$$

[C.-Pouchin]

$$\mathrm{Irr}\,(\mathbb{C}(q)\,FTL_{d,n}(q)) \longleftrightarrow \left\{ \begin{array}{l} d\text{-partitions } \lambda = (\lambda^{(1)}, \dots, \lambda^{(d)}) \text{ of } n \\ \text{such that the Young diagram of} \\ \text{each } \lambda^{(i)} \text{ has at most 2 columns} \end{array} \right\}$$

Basis : Work in progress!