Decomposition matrices for cyclotomic Hecke algebras

Maria Chlouveraki

University of Edinburgh

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Complex reflection groups

A complex reflection group W is a finite group of matrices with coefficients in a finite abelian extension K of $\mathbb{Q}$ generated by *pseudo-reflections*.

If $K = \mathbb{Q}$, then W is a Weyl group.

Shephard-Todd classification (1954)

The irreducible complex reflection groups are:

- the groups of the infinite series $G(de, e, r)$

  (with $G(d, 1, r) \cong \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_r$);
- the exceptional groups G_4, G_5, $\ldots$, G_{37}.

Hecke algebras of complex reflection groups

Let W be a complex reflection group.

The group W has a presentation given by:

- generators: S
- relations:
  - braid relations;
  - $(s-1)(s-\zeta_{e_s})\cdots(s-\zeta_{e_s}^{e_s-1})=0.$ $[\zeta_{e_s} := \exp(2\pi i/e_s)]$

Example:

$$G := G(3,1,2) = \langle\, s,t \,|\, stst = tsts,\ s^3 = 1,\ t^2 = 1 \,\rangle.$$

Let q be an indeterminate and let $A := \mathbb{Z}_K[q, q^{-1}]$.

The cyclotomic Hecke algebra $\mathcal{H}_q(W)$ has a presentation given by:

- generators: $(T_s)_{s\in S}$
- relations:
  - braid relations;
  - $(T_s - 1q^{m_{s,0}})(T_s - \zeta_{e_s} q^{m_{s,1}}) \cdots (T_s - \zeta_{e_s}^{e_s-1} q^{m_{s,e_s-1}}) = 0.$

Example: $G = G(3,1,2)$

$$\mathcal{H}_q(G) = \left\langle \; T_s, T_t \; \middle| \; \begin{array}{l} T_sT_tT_sT_t = T_tT_sT_tT_s, \\ (T_s - q^{m_{s,0}})(T_s - \zeta_3 q^{m_{s,1}})(T_s - \zeta_3^2 q^{m_{s,2}}) = 0, \\ (T_t - q^{m_{t,0}})(T_t + q^{m_{t,1}}) = 0 \end{array} \right\rangle .$$

Schur elements of Hecke algebras

(We make some assumptions.)

The algebra $K(q)\mathcal{H}_q(W)$ is semisimple. By Tits's deformation theorem, we have a bijection:

$$\begin{array}{ccc} \mathrm{Irr}(K(q)\mathcal{H}_q(W)) & \leftrightarrow & \mathrm{Irr}(W) \\ \chi_q & \mapsto & \chi. \end{array}$$

Moreover, there exists a "canonical" symmetrizing form $t : \mathcal{H}_q(W) \to A$, such that

$$t = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{s_\chi} \chi_q$$

where s_χ is the Schur element of $\mathcal{H}_q(W)$ associated with χ.

We have that s_χ belongs to $A = \mathbb{Z}_K[q, q^{-1}]$ and it is a product of cyclotomic polynomials over K.

Definition

We define a_χ to be the smallest non-negative integer such that

$$q^{a_\chi} s_\chi \in \mathbb{Z}_K[q].$$

Example: If $s_\chi = q^{-1} + 2 + q$, then $a_\chi = 1$.

The decomposition matrix

Let

$$\theta : A \to \mathbb{C}, \quad q \mapsto \xi$$

be a ring homomorphism. Set $\mathcal{H}_\xi := \mathbb{C} \otimes_A \mathcal{H}_q(W)$.

Theorem (Geck-Pfeiffer)

The algebra $\mathcal{H}_\xi$ is semisimple if and only if $\theta(s_\chi) \neq 0$ for all $\chi \in \mathrm{Irr}(W)$.

We have a well-defined decomposition map

$$d_\theta : R_0(K(q)\mathcal{H}_q(W)) \to R_0(\mathcal{H}_\xi)$$

with corresponding decomposition matrix

$$D_\theta = (\, [E : M] \,)_{E \in \mathrm{Irr}(W),\, M \in \mathsf{Irr}(\mathcal{H}_\xi)} \,.$$

Basic sets

Definition (Geck-Rouquier)

We say that $\mathcal{H}_q(W)$ admits a canonical basic set $\mathcal{B}^{\mathrm{can}} \subset \mathrm{Irr}(W)$ with respect to $\theta : A \to \mathbb{C}$ if there exists a bijection

$$\begin{array}{ccc} \mathrm{Irr}(\mathcal{H}_\xi) & \leftrightarrow & \mathcal{B}^{\mathrm{can}} \\ M & \mapsto & E_M \end{array}$$

such that

- $[E_M : M] = 1$, and
- if $[E : M] \neq 0$, then either $E = E_M$ or $a_E > a_{E_M}$.

If $\mathcal{H}_q(W)$ admits a canonical basic set $\mathcal{B}^{\mathrm{can}}$ with respect to θ, then the decomposition matrix D_θ has the following form:

$$D_\theta = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}}_{\mathrm{Irr}(\mathcal{H}_\xi)} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \mathcal{B}^{\mathrm{can}} \\ \\ \\ \\ \end{matrix} \quad \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} \mathrm{Irr}(W)$$

Theorem

The algebra $\mathcal{H}_q(W)$ admits a canonical basic set with respect to any specialization $\theta : A \to \mathbb{C}$, if

1. W is a Weyl group;
   [*Geck-Rouquier, Geck, Geck-Jacon, C.-Jacon*]
2. W is a complex reflection group of type $G(d, 1, r)$;
   [*Dipper-James-Murphy, Geck-Rouquier, Ariki, Uglov, Jacon*]
3. W is a complex reflection group of type $G(de, e, r)$
   (for a certain choice of paremeters);
   [*Genet-Jacon*]
4. $W \in \{G_4, G_5, G_8, G_9, G_{10}, G_{12}, G_{16}, G_{20}, G_{22}\}$
   (for certain choices of paremeters).
   [*C.-Miyachi*]

In the last case, we have been also able to show that there exists a subset $\mathcal{B}^{\mathrm{opt}} \subset \mathrm{Irr}(W)$ such that the decomposition matrix D_θ has the following form:

$$D_\theta = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}}_{\mathrm{Irr}(\mathcal{H}_\xi)} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \mathcal{B}^{\mathrm{opt}} \\ \\ \\ \\ \end{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} \mathrm{Irr}(W)$$