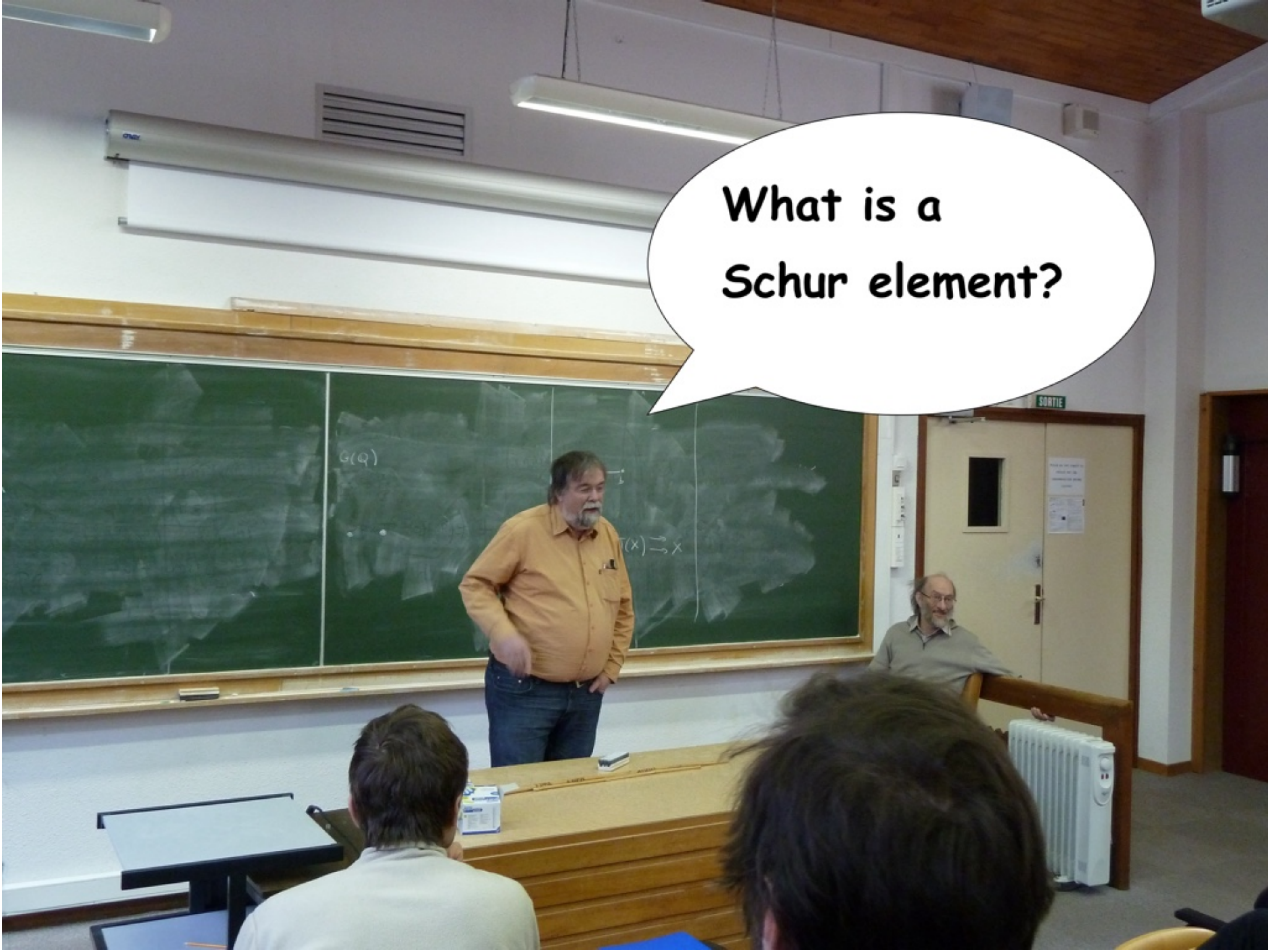
Schur elements for Hecke algebras

Maria Chlouveraki

Université de Versailles - St Quentin

Finite Chevalley groups, reflection groups and braid groups

A conference in honour of Professor Jean Michel


What is a
Schur element?

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- A be an R-algebra, free and finitely generated as an R-module ;
- K be a splitting field for A.

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A *symmetrising trace* on the algebra A is a linear map $\tau : A \to R$ such that

1. $\tau(ab) = \tau(ba)$ for all $a, b \in A$, and
2. the map $\widehat{\tau} : A \to \mathrm{Hom}_R(A, R)$, $a \mapsto (x \mapsto \tau(ax))$ is an isomorphism of A-bimodules.

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Example

Let G be a finite group. The linear map $\tau : \mathbb{Z}[G] \to \mathbb{Z}$, $\sum_{g\in G} r_g g \mapsto r_1$ is the *canonical symmetrising trace* on $\mathbb{Z}[G]$.

Let E be a simple KA-module and let χ_E be the corresponding character. The map τ can be extended to KA by extension of scalars. We have:

$$\widehat{\tau}^{-1}(\chi_E) \in Z(KA).$$

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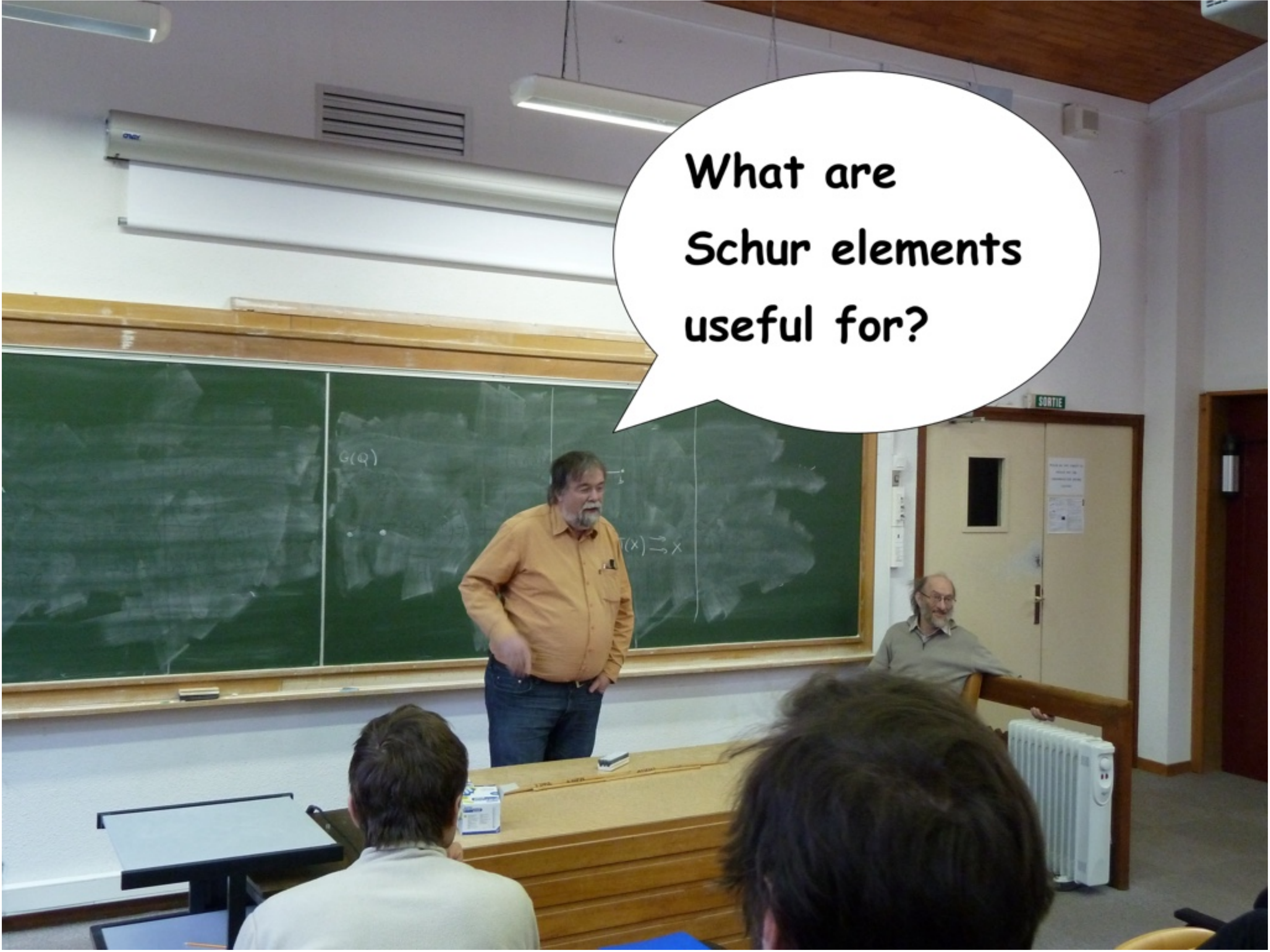
Example

Let G be a finite group and let τ be the canonical symmetrising trace on $A := \mathbb{Z}[G]$. Let K be an algebraically closed field of characteristic 0, and let $E \in \mathrm{Irr}(KA)$. We have $s_E = |G|/\chi_E(1) \in \mathbb{Z}_K \cap \mathbb{Q} = \mathbb{Z}$.
We have thus shown that $\chi_E(1)$ divides $|G|$.


What are Schur elements useful for?

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   $$\tau = \sum_{E \in \mathrm{Irr}(KA)} \frac{1}{s_E} \chi_E.$$
   - The blocks of A are the non-empty subsets B of $\mathrm{Irr}(KA)$ that are minimal with respect to the property:
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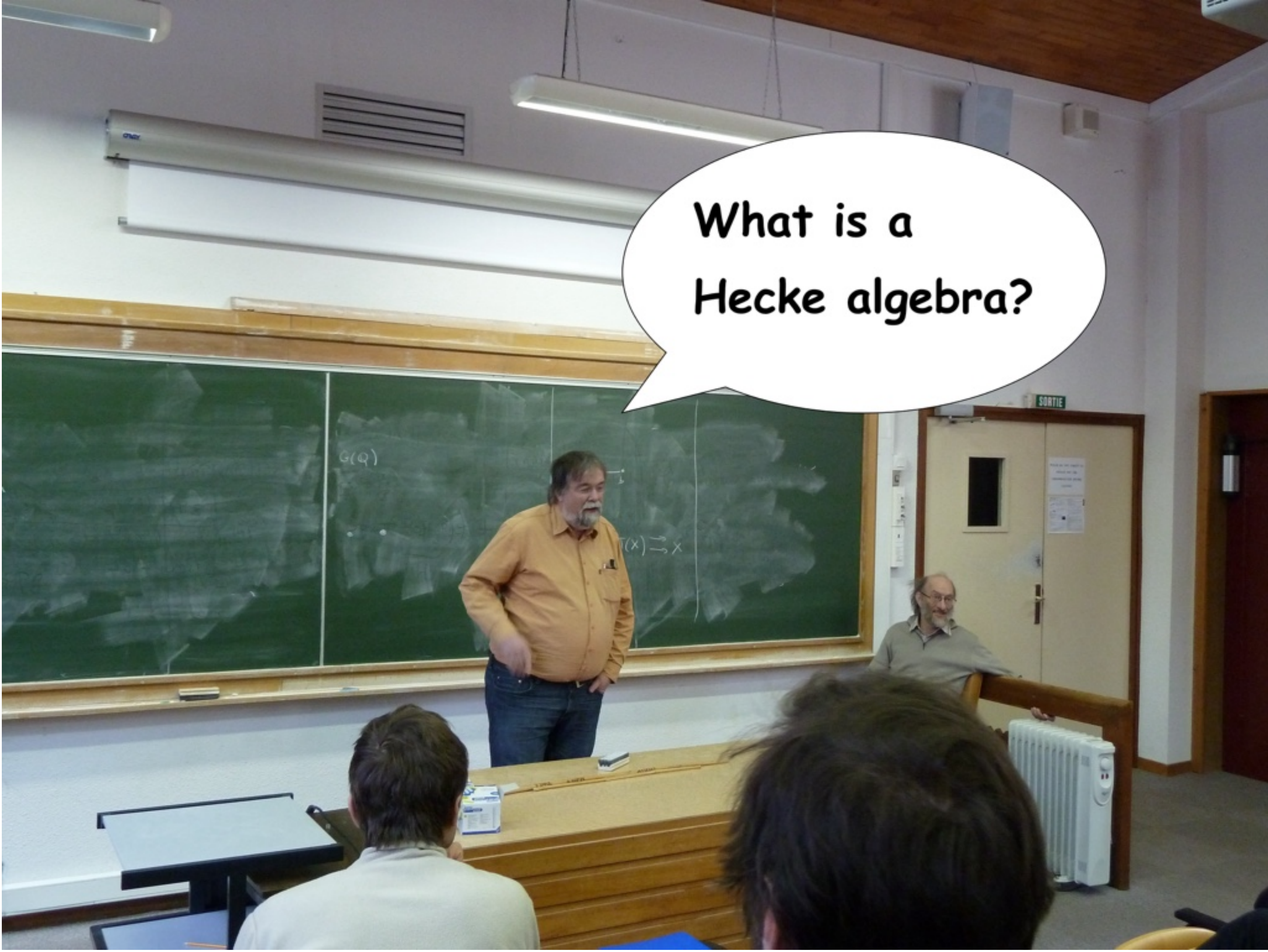
   - LA is semisimple if and only if $\theta(s_E) \neq 0$ for all $E \in \mathrm{Irr}(KA)$.
   - If $\theta(s_E) \neq 0$ for some $E \in \mathrm{Irr}(KA)$, then E forms a block of defect 0.


What is a
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SORTIE

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Theorem (Shephard-Todd)

Let $W \subset \mathrm{GL}(V)$ be an irreducible complex reflection group (*i.e.*, W acts irreducibly on V). Then one of the following assertions is true:

- $(W, V) \cong (\mathfrak{S}_r, \mathbb{C}^{r-1})$.
- $(W, V) \cong (G(de, e, r), \mathbb{C}^r)$, where $G(de, e, r)$ is the group of all $r \times r$ monomial matrices whose non-zero entries are de-th roots of unity, while the product of all non-zero entries is a d-th root of unity.
- (W, V) is isomorphic to one of the 34 exceptional groups G_n, $n = 4, \ldots, 37$.

Every complex reflection group W has a presentation "à la Coxeter" given by generators and relations. The *generic Hecke algebra* $\mathcal{H}(W)$ of W can be viewed as a deformation of the group algebra of W.

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Example

The generic Hecke algebra of

$$G_6 = \langle\, s, t \mid ststst = tststs,\ s^2 = t^3 = 1 \,\rangle$$

is

$$\mathcal{H}(G_6) = \left\langle\ T_s,\ T_t\ \middle|\ \begin{array}{l} T_sT_tT_sT_tT_sT_t = T_tT_sT_tT_sT_tT_s, \\ (T_s - u_{s,0})(T_s - u_{s,1}) = 0, \\ (T_t - u_{t,0})(T_t - u_{t,1})(T_t - u_{t,2}) = 0 \end{array}\ \right\rangle.$$

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Set $\zeta_d := \exp(2\pi i/d)$. The algebra $\mathcal{H}(G_6)$ specialises to $\mathbb{Z}[G_6]$ when

$$u_{s,0} \mapsto 1, u_{s,0} \mapsto -1, u_{t,0} \mapsto 1, u_{t,1} \mapsto \zeta_3, u_{t,2} \mapsto \zeta_3^2.$$

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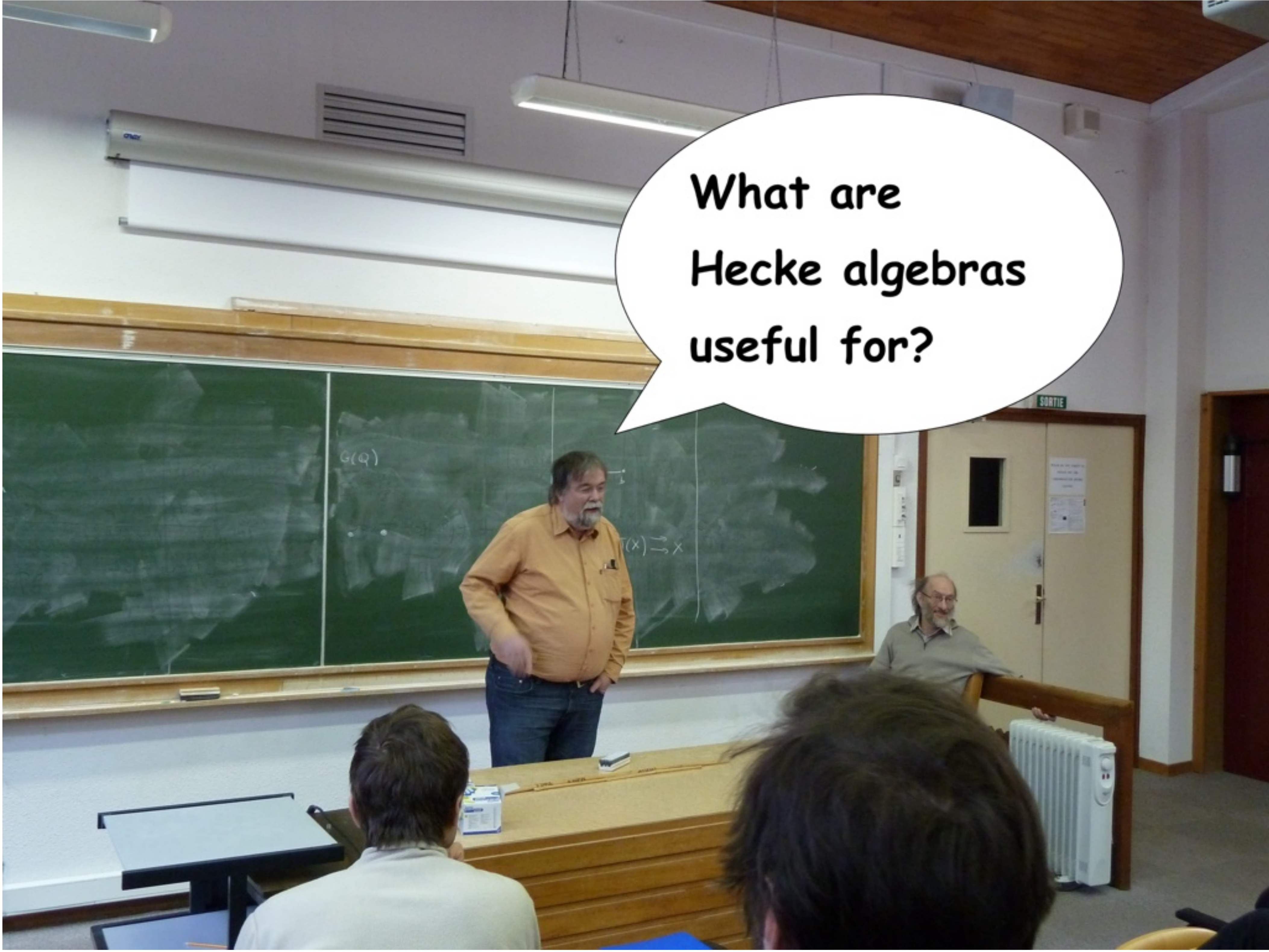
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1. Representation theory of finite reductive groups.

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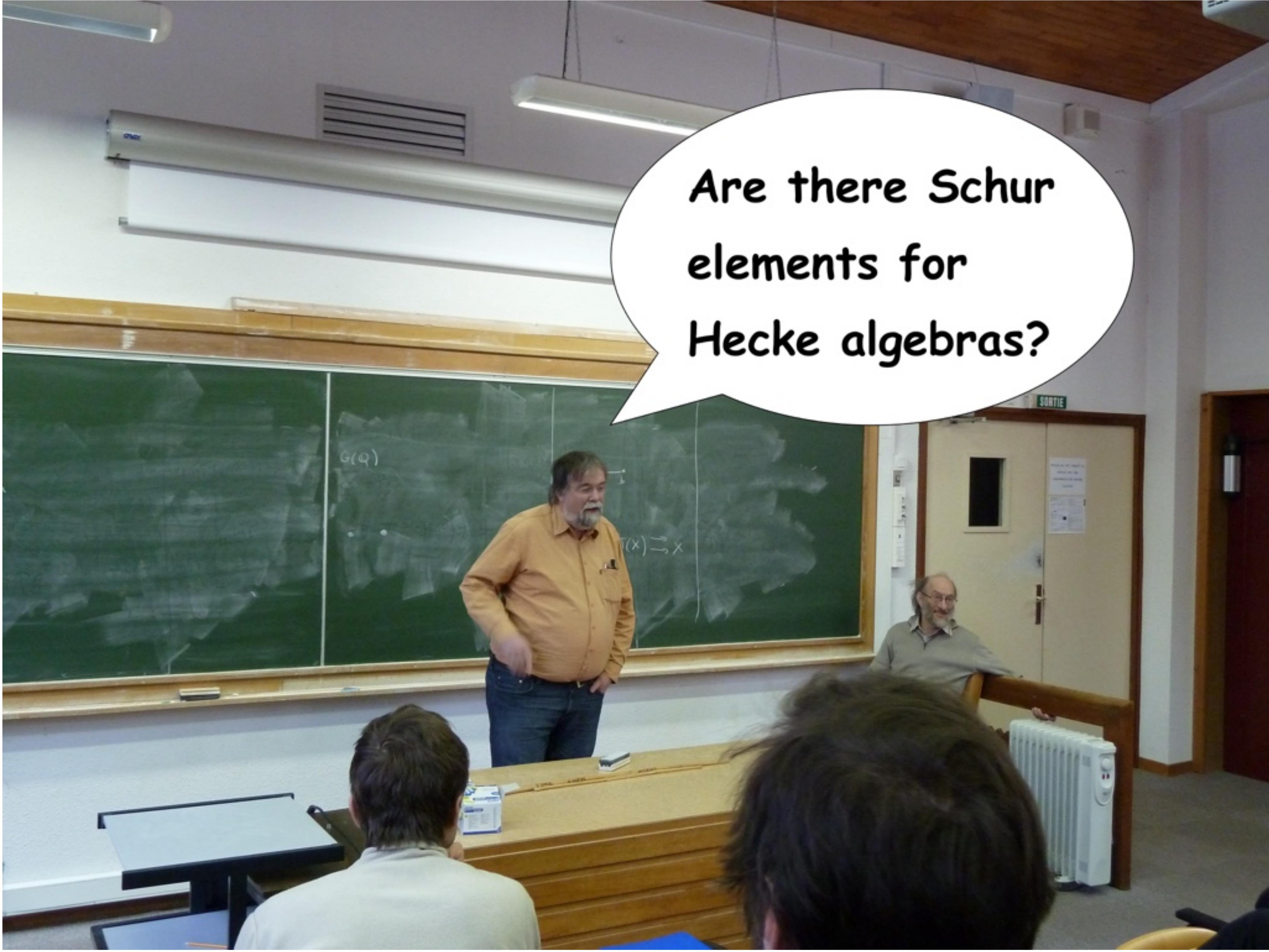
Are there Schur elements for Hecke algebras?
SORTIE

Conjectures (Broué-Malle-Michel-Rouquier)

1. **Freeness:** The algebra $\mathcal{H}(W)$ is a free $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-module of rank $|W|$.
2. **Trace:** There exists a canonical symmetrising trace τ on $\mathcal{H}(W)$ that satisfies certain canonicality conditions; the map τ specialises to the canonical symmetrising trace on the group algebra of W when $u_{s,j} \mapsto \zeta_{e_s}^{j}$.

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The Freeness Conjecture is verified for:

- all finite Coxeter groups ;
- $G(de,e,r)$ (Ariki-Koike, Broué-Malle-Michel) ;
- all exceptional groups except for $G_{17},\ldots,G_{21}$ (Chavli, Marin, Marin-Pfeiffer).

The Trace Conjecture is verified for:

- all finite Coxeter groups ;
- $G(de,e,r)$ (BMM, Bremke-Malle, Malle-Mathas, Geck-Iancu-Malle) ;
- the exceptional groups G_4, G_{12}, G_{22} and G_{24} (Malle-Michel).

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Theorem (Benard, Bessis)

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We can always find $N_W \in \mathbb{Z}_{>0}$ such that if we take

$$u_{s,j} = \zeta_{e_s}^{j} v_{s,j}^{N_W}$$

and set $\mathbf{v} := (v_{s,j})_{s,j}$, then the $K(\mathbf{v})$-algebra $K(\mathbf{v})\mathcal{H}(W)$ is split semisimple.

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Example

For $W = G_6$, we have $K = \mathbb{Q}(\zeta_{12})$ and we can take $N_W = 2$.

$$\mathcal{H}(G_6) = \left\langle \; T_s, \; T_t \; \middle| \; \begin{array}{l} T_sT_tT_sT_tT_sT_t = T_tT_sT_tT_sT_tT_s, \\ (T_s - v_{s,0}^2)(T_s + v_{s,1}^2) = 0, \\ (T_t - v_{t,0}^2)(T_t - \zeta_3 v_{t,1}^2)(T_t - \zeta_3^2 v_{t,2}^2) = 0 \end{array} \; \right\rangle$$

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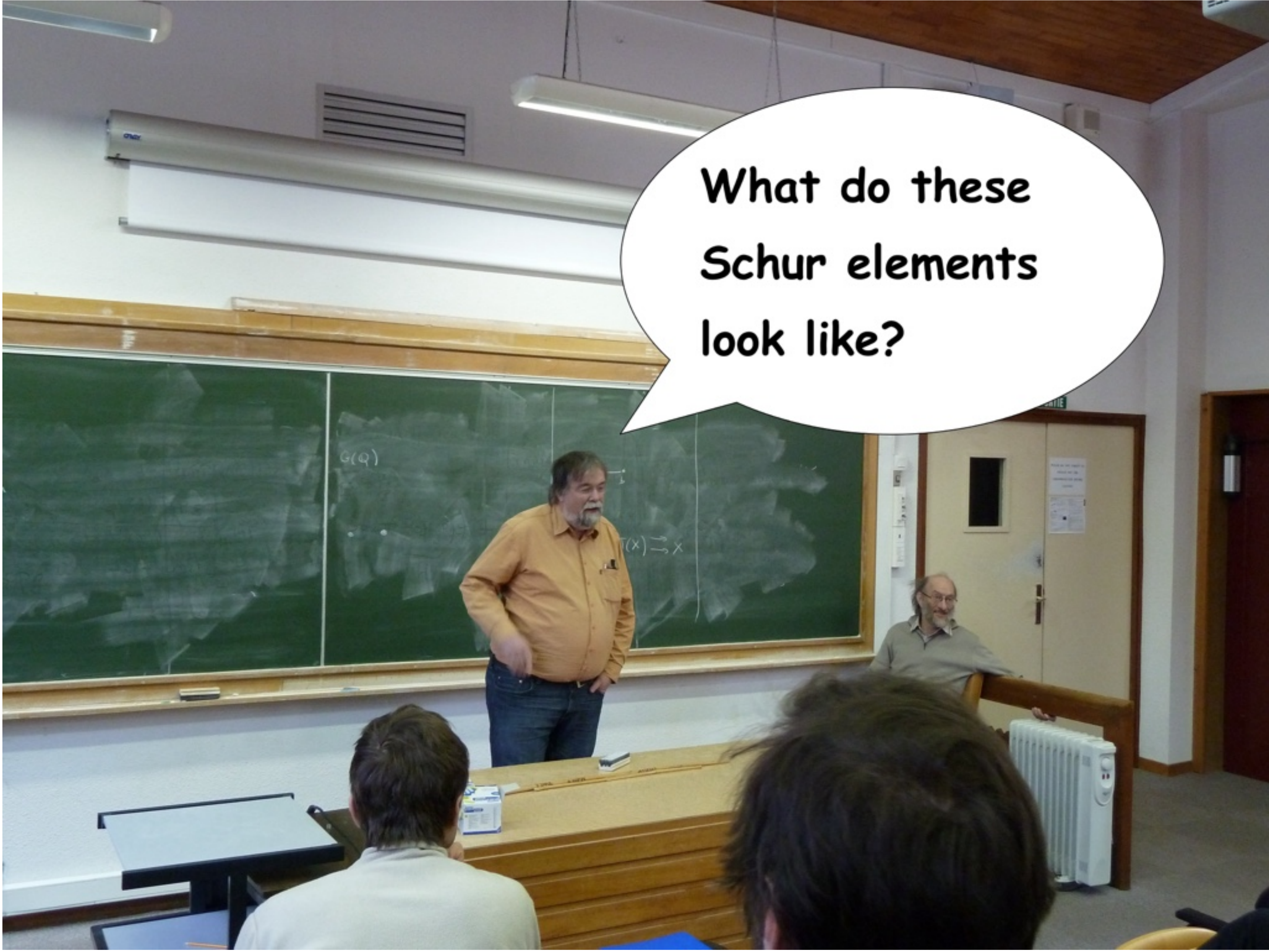
Tits's Deformation Theorem $\quad \Rightarrow \quad \mathrm{Irr}(K(\mathbf{v})\mathcal{H}(W)) \leftrightarrow \mathrm{Irr}(W).$

The Schur elements of $\mathcal{H}(W)$ have been explicitly calculated for

- all finite Coxeter groups :
  - for type A_n by Steinberg,
  - for type B_n by Hoefsmit,
  - for type D_n by Benson and Gay,
  - for dihedral groups $I_2(m)$ by Kilmoyer and Solomon,
  - for F_4 by Lusztig,
  - for E_6 and E_7 by Surowski,
  - for E_8 by Benson,
  - for H_3 by Lusztig,
  - for H_4 by Alvis and Lusztig ;
- $G(d, 1, r)$ by Geck-Iancu-Malle and Mathas ;
- $G(2d, 2, 2)$ by Malle ;
- for the non-Coxeter exceptional complex reflection groups by Malle.

With the use of Clifford theory, we obtain the Schur elements for $G(de, e, r)$ from those of $G(de, 1, r)$ when $r > 2$ or $r = 2$ and e is odd. The Schur elements for $G(de, e, 2)$ when e is even derive from those of $G(de, 2, 2)$.


What do these Schur elements look like?

Theorem (C.)

Let $E \in \mathrm{Irr}(W)$. The Schur element s_E is an element of $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ of the form

$$s_E = \xi_E N_E \prod_{i \in I_E} \Psi_{E,i}(M_{E,i})$$

where

- ξ_E is an element of $\mathbb{Z}_K$,
- $N_E = \prod_{s,j} v_{s,j}^{b_{s,j}}$ is a monomial in $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ with $\sum_{j=0}^{e_s-1} b_{s,j} = 0$ for all s,
- I_E is an index set,
- $(\Psi_{E,i})_{i \in I_E}$ is a family of K-cyclotomic polynomials in one variable,
- $(M_{E,i})_{i \in I_E}$ is a family of monomials in $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ such that if $M_{E,i} = \prod_{s,j} v_{s,j}^{a_{s,j}}$, then $\gcd(a_{s,j}) = 1$ and $\sum_{j=0}^{e_s-1} a_{s,j} = 0$ for all s.

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- I_E is an index set,
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This is the factorisation of s_E into irreducible factors. The monomials $(M_{E,i})_{i \in I_E}$ are unique up to inversion.

Example

Take $K = \mathbb{Q}$ and $\Phi_4(x) = x^2 + 1$. Then

$$\Phi_4(ab^{-1}) = a^2 b^{-2} + 1 = a^2 b^{-2}(1 + a^{-2} b^2) = a^2 b^{-2} \Phi_4(a^{-1} b).$$

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$gap > W := ComplexReflectionGroup(6);;$

$gap > H := Hecke(W, [[a^2, -b^2], [c^2, E(3) * d^2, E(3)^2 * e^2]]);;$

$gap > FactorizedSchurElement(H, [[1, 0]]);$

$P_4(ab^{-1})P_3''P_6'(cd^{-1})P_3'P_6''(ce^{-1})P_4P_{12}'''(ab^{-1}cd^{-1})P_4P_{12}''''(ab^{-1}ce^{-1})$
$P_4(ab^{-1}c^2d^{-1}e^{-1})$

where $P_4 = x^2 + 1$, $P_3' = (x - \zeta_3)$, $P_3'' = (x - \zeta_3^2)$, etc.


And for G(d,1,r) ?

For $W = G(d,1,r) \cong (\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_r \cong (\mathbb{Z}/d\mathbb{Z})^r \rtimes \mathfrak{S}_r$, $\mathcal{H}(W)$ is generated by elements

$$T_0, T_1, \ldots, T_{r-1}$$

satisfying the braid relations of type B_r:

and the extra relations:

$$(T_0 - Q_0)(T_0 - Q_1)\cdots(T_0 - Q_{d-1}) = 0 \quad \text{and} \quad (T_i - q_0)(T_i - q_1) = 0$$

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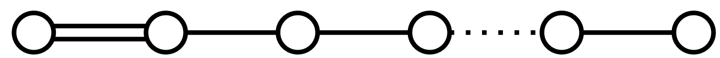

and the extra relations:

$$(T_0 - \mathrm{Q}_0)(T_0 - \zeta_d \mathrm{Q}_1) \cdots (T_0 - \zeta_d^{d-1}\mathrm{Q}_{d-1}) = 0 \quad \text{and} \quad (T_i - \mathrm{q}_0)(T_i + \mathrm{q}_1) = 0$$

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The algebra $\mathcal{H}(W)$ is split semisimple over the field $K := \mathbb{Q}(Q_0, Q_1, \ldots, Q_{d-1}, q)$.

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We have

$$\mathrm{Irr}(\mathcal{K}\mathcal{H}(W)) \leftrightarrow \mathrm{Irr}(W) \leftrightarrow \{d\text{-partitions of } r\}.$$

The Schur elements of Ariki–Koike algebras have been independently determined by Geck-Iancu-Malle and Mathas. They belong to $\mathbb{Z}[Q_0^{\pm 1}, Q_1^{\pm 1}, \ldots, Q_{d-1}^{\pm 1}, q^{\pm 1}]$.

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Theorem (C.-Jacon)

Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ be a d-partition of r. Then

$$s_{\boldsymbol{\lambda}} = (-1)^{r(d-1)} q^{-m_{\boldsymbol{\lambda}}} (q-1)^{-r} \prod_{0 \leq s \leq d-1} \prod_{(i,j) \in [\lambda^{(s)}]} \prod_{0 \leq t \leq d-1} \left(q^{h_{i,j}^{\lambda^{(s)},\lambda^{(t)}}} Q_s Q_t^{-1} - 1\right)$$

where

- $m_{\boldsymbol{\lambda}} \in \mathbb{N}$,
- $[\lambda^{(s)}] = \{(i,j) \mid i \geq 1, 1 \leq j \leq \lambda_i^{(s)}\}$ is the set of nodes of $\lambda^{(s)}$,
- $h_{i,j}^{\lambda^{(s)},\lambda^{(t)}} := \lambda_i^{(s)} - i + \lambda_j^{(t)'} - j + 1$ is the *generalised hook length* of the node (i,j) with respect to $(\lambda^{(s)}, \lambda^{(t)})$.

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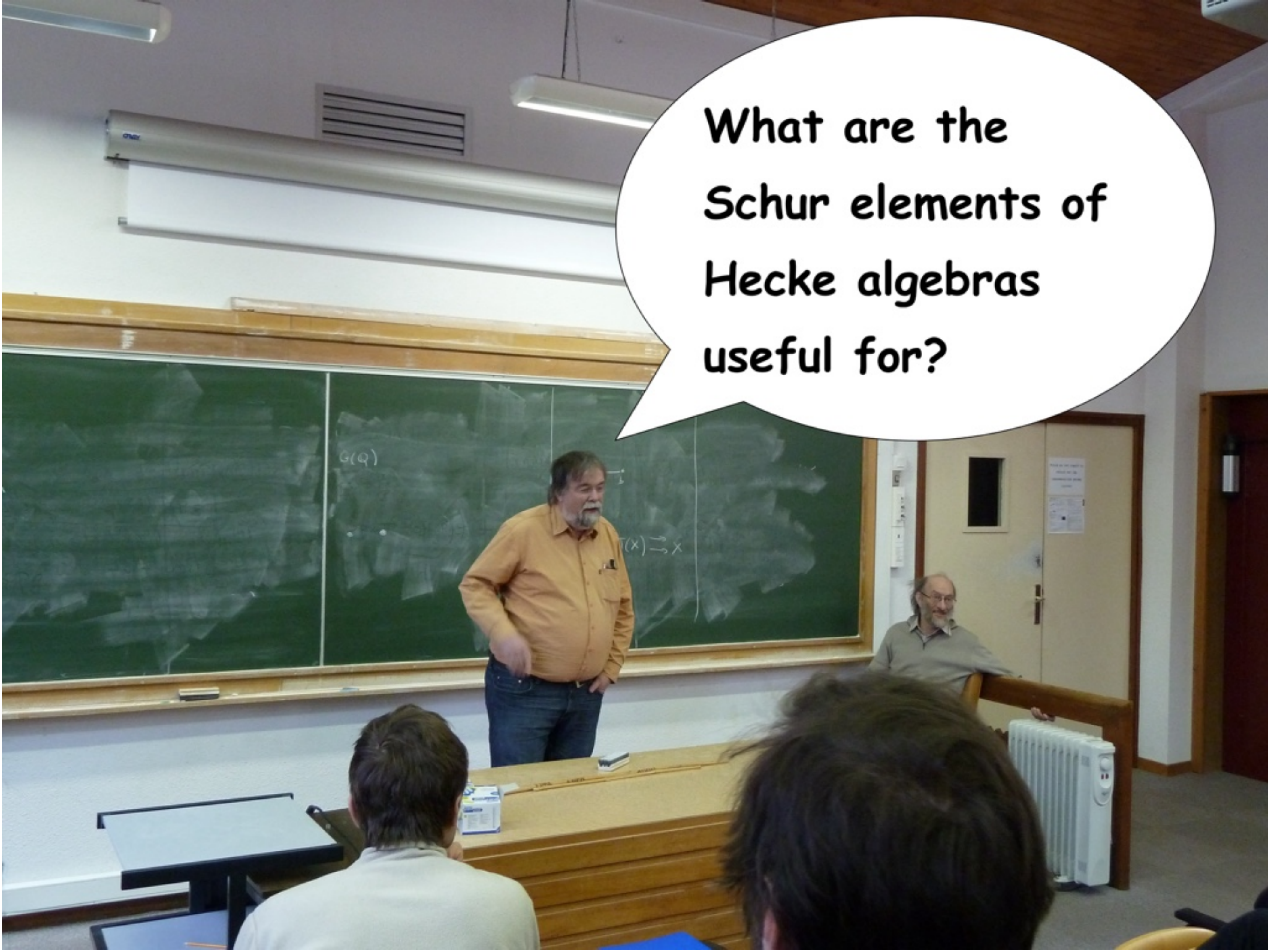
CHEVIE (C.-Michel)

SchurModels, SchurData, FactorizedSchurElement, FactorizedSchurElements.


What are the Schur elements of Hecke algebras useful for?

Consider $\mathcal{H}(W)$ defined over $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$, and let q be an indeterminate.
A *cyclotomic specialisation* is a $\mathbb{Z}_K$-algebra homomorphism

$$\varphi : \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}] \to \mathbb{Z}_K[q, q^{-1}], \;\; v_{s,j} \mapsto q^{m_{s,j}} \, ,$$

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We define $a : \mathrm{Irr}(W) \to \mathbb{Z}$ and $A : \mathrm{Irr}(W) \to \mathbb{Z}$ such that

$$a(E) := -\mathrm{Valuation}_q(\varphi(s_E)) \quad \text{and} \quad A(E) := -\mathrm{Degree}_q(\varphi(s_E)).$$

Example

If $\varphi(s_E) = q^{-1}\Phi_5(q) = q^{-1} + 1 + q + q^2 + q^3$, then $a(E) = 1$ and $A(E) = -3$.

Canonical Basic Sets

Let $\theta : \mathbb{Z}_K[q, q^{-1}] \to K(\eta)$, $q \mapsto \eta \in \mathbb{C}^*$ be a ring homomorphism. Let $\mathcal{H}_\theta(W)$ be the algebra obtained as a specialisation of $\mathcal{H}_\varphi(W)$ via θ.

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If $K(\eta)\mathcal{H}_\theta(W)$ is not semisimple, we obtain a decomposition matrix D_θ.
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Canonical basic sets are proved to exist and explicitly described for:

- all finite Coxeter groups:
  - existence by Geck-Rouquier, Geck-Jacon, Geck ;
  - description for type A_n by Geck, for type B_n by Jacon, for type D_n by Geck and Jacon, for all remaining groups by Geck, Lux and Müller ;
- for $G(d, 1, r)$ by Geck and Jacon ;
- for $G(de, e, r)$ by Genet-Jacon, C.-Jacon ;
- for some exceptional cases by C.-Miyachi.

Families of Characters

The *Rouquier families* are the blocks of $\mathcal{H}_\varphi(W)$ over the *Rouquier ring* :

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These are the non-empty subsets B of $\mathrm{Irr}(W)$ that are minimal with respect to the property :

$$\sum_{E\in B} \frac{1}{\varphi(s_E)}\varphi(\chi_E)(h) \in \mathcal{R}_K(q) \quad \forall h \in \mathcal{H}_\varphi(W).$$

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and

$\Psi_{E,i}(q^{\sum a_{s,j} m_{s,j}})$ is a product of K-cyclotomic polynomials unless $\sum_{s,j} a_{s,j} m_{s,j} = 0$.

Example

Take $a = q^4$, $b = q$. We have $\Phi_4(ab^{-1}) = \Phi_4(q^3) = q^6 + 1 = \Phi_4(q)\Phi_{12}(q)$.

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Now, if $\varphi : v_{s,j} \mapsto q^{m_{s,j}}$ is a cyclotomic specialisation such that

1. the integers $m_{s,j}$ belong to no essential hyperplane, then the Rouquier families of $\mathcal{H}_\varphi(W)$ are called *Rouquier families associated with no essential hyperplane*.
2. the integers $m_{s,j}$ belong to a unique essential hyperplane H, then the Rouquier families of $\mathcal{H}_\varphi(W)$ are called *Rouquier families associated with H*.

The above notions are well-defined because of the following theorem:

Example

Take $a = q^4$, $b = q$. We have $\Phi_4(ab^{-1}) = \Phi_4(q^3) = q^6 + 1 = \Phi_4(q)\Phi_{12}(q)$.
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Theorem (C.)

Let $\varphi : v_{s,j} \mapsto q^{m_{s,j}}$ be a cyclotomic specialisation. The Rouquier families of $\mathcal{H}_\varphi(W)$ are unions of the Rouquier families associated with the essential hyperplanes that the $m_{s,j}$ belong to and they are minimal with respect to this property.

Example

$$\mathcal{H}(G_6) = \left\langle \; T_s,\; T_t \;\middle|\; \begin{array}{l} T_s T_t T_s T_t T_s T_t = T_t T_s T_t T_s T_t T_s, \\ (T_s - u_{s,0})(T_s - u_{s,1}) = 0, \\ (T_t - u_{t,0})(T_t - u_{t,1})(T_t - u_{t,2}) = 0 \end{array} \right\rangle.$$

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no condition

$\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$c_1 - c_2 = 0$

$\{\phi_{1,4}, \phi_{1,8}\}$, $\{\phi_{1,10}, \phi_{1,14}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,3}, \phi_{2,1}, \phi'_{2,5}\}$

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$a_0 - a_1 - 2c_0 + c_1 + c_2 = 0$

$\{\phi_{1,6}, \phi''_{2,5}, \phi_{2,7}, \phi_{3,4}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 + c_0 - 2c_1 + c_2 = 0$

$\{\phi_{1,10}, \phi''_{2,3}, \phi'_{2,5}, \phi_{3,4}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 + c_0 + c_1 - 2c_2 = 0$

$\{\phi_{1,14}, \phi'_{2,3}, \phi_{2,1}, \phi_{3,4}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$

$a_0 - a_1 - c_0 - c_1 + 2c_2 = 0$

$\{\phi_{1,8}, \phi'_{2,3}, \phi_{2,1}, \phi_{3,2}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$

$a_0 - a_1 - c_0 + 2c_1 - c_2 = 0$

$\{\phi_{1,4}, \phi''_{2,3}, \phi'_{2,5}, \phi_{3,2}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 + 2c_0 - c_1 - c_2 = 0$

$\{\phi_{1,0}, \phi''_{2,5}, \phi_{2,7}, \phi_{3,2}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 - c_0 + c_1 = 0$

$\{\phi_{1,4}, \phi_{1,6}, \phi'_{2,3}, \phi_{2,1}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$

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$a_0 - a_1 = 0$

$\{\phi_{1,0}, \phi_{1,6}\}$, $\{\phi_{1,4}, \phi_{1,10}\}$, $\{\phi_{1,8}, \phi_{1,14}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$, $\{\phi_{3,2}, \phi_{3,4}\}$

Example

$$\mathcal{H}_\varphi(G_6) = \left\langle \; T_s,\; T_t \;\middle|\; \begin{array}{l} T_sT_tT_sT_tT_sT_t = T_tT_sT_tT_sT_tT_s, \\ (T_s - q^{2a_0})(T_s + q^{2a_1}) = 0, \\ (T_t - q^{2c_0})(T_t - \zeta_3 q^{2c_1})(T_t - \zeta_3^2 q^{2c_2}) = 0 \end{array} \right\rangle .$$

CHEVIE (C.-Michel)

For all exceptional complex reflection groups: *Rouquierblockdata.g*.

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$$\mathcal{H}_\varphi(G_6) = \left\langle \; T_s,\; T_t \;\middle|\; \begin{array}{l} T_sT_tT_sT_tT_sT_t = T_tT_sT_tT_sT_tT_s, \\ (T_s - q^{2a_0})(T_s + q^{2a_1}) = 0, \\ (T_t - q^{2c_0})(T_t - \zeta_3 q^{2c_1})(T_t - \zeta_3^2 q^{2c_2}) = 0 \end{array} \right\rangle .$$

CHEVIE (C.-Michel)

For all exceptional complex reflection groups: *Rouquierblockdata.g*.

CHEVIE (C.)

For the Ariki-Koike algebras: *RBAK.g*.


Anything else to add?

Theorem (Lusztig)

The functions a and A are constant on the families of characters.

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The function $a + A$ is constant on the Rouquier families of any cyclotomic Hecke algebra associated with a complex reflection group.

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Theorem (C.)

The functions a and A are constant on the Rouquier families of any cyclotomic Hecke algebra associated with an exceptional complex reflection group.

A cyclotomic polynomial in one variable has valuation 0.

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We have

$$\varphi : \Psi_{E,i}(\prod_{s,j} v_{s,j}^{a_{s,j}}) \mapsto \Psi_{E,i}(q^{\sum a_{s,j} m_{s,j}}).$$

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We distinguish two cases:

- If $\sum a_{s,j} m_{s,j} \geq 0$, we add 0 to $a(E)$.
- If $\sum a_{s,j} m_{s,j} < 0$, we add $-\mathrm{degree}(\Psi_{E,i}) \cdot (\sum a_{s,j} m_{s,j})$ to $a(E)$.

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CHEVIE (C.)

For all excpetional complex reflection groups: *DegVal.g*.

where $\xi_\chi \in \mathbb{Z}_K$, $b_\chi \in \mathbb{Z}$, C_χ is a set of K-cyclotomic polynomials and $n_{\chi,\psi} \in \mathbb{N}$. If $\phi : v \mapsto y^n$ ($n \in \mathbb{Z}$) is a cyclotomic specialization, then

- $a_{\chi_\phi} = n \cdot \mathrm{val}_v(s_\chi(v))$.
- $A_{\chi_\phi} = n \cdot \deg_v(s_\chi(v))$.

Therefore, in order to verify Theorem 6.1 for W, it suffices to check whether the degree and the valuation of the generic Schur elements remain constant on the Rouquier blocks associated with no essential hyperplane. Note that the generic Schur elements coincide with the Schur elements of the "spetsial" cyclotomic Hecke algebra and the Rouquier blocks associated with no essential hyperplane coincide with its Rouquier blocks.

We can easily create an algorithm which returns "true" if the degree and the valuation of the

Theorem 6.1 holds for W.

Acknowledgments

I would like to thank Jean Michel for making my algorithm look better and run faster. I would also like to thank the Ecole Polytechnique Fédérale de Lausanne for its financial support.

Appendix A

Definition A.1. Let $\bar{A}$ be a subalgebra of A free and of finite rank as an $\mathcal{O}$-module. We say that $\bar{A}$ is a symmetric subalgebra of A, if it satisfies the following two conditions:

1. $\bar{A}$ is free (of finite rank) as an $\mathcal{O}$-module and the restriction $\mathrm{Res}^A_{\bar{A}}(t)$ of the form t to $\bar{A}$ is a

Thank you for listening (everyone for the past hour, Jean for the past ten years)!