TWO LEMMAS :
one seemingly true and one seemingly false

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Lemma T

Let R be an integrally closed domain and let F be its field of fractions. Let $\mathfrak{p}$ be a prime ideal of R. Then

$$(R[x])_{\mathfrak{p}R[x]} \cap F[x] = R_{\mathfrak{p}}[x].$$

Lemma F

Let R be an integrally closed domain and $f(x) = \sum_i a_i x^i$, $g(x) = \sum_j b_j x^j$ be two polynomials in $R[x]$. If there exists an element $c \in R$ such that all the coefficients of $f(x)g(x)$ belong to cR, then all the products $a_i b_j$ belong to cR.

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(V1) $v(xy) = v(x) + v(y)$ for $x, y \in F$.

(V2) $v(x + y) \geqslant \min(v(x), v(y))$ for $x, y \in F$.

(V3) $v(0) = \infty$.

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Let R be an integrally closed domain and $f(x) = \sum_i a_i x^i$, $g(x) = \sum_j b_j x^j$ be two polynomials in $R[x]$. If there exists an element $c \in R$ such that all the coefficients of $f(x)g(x)$ belong to cR, then all the products $a_i b_j$ belong to cR.

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Let R be an integrally closed domain and let F be its field of fractions. Let $\mathfrak{p}$ be a prime ideal of R. Then

$$(R[x])_{\mathfrak{p}R[x]} \cap F[x] = R_{\mathfrak{p}}[x].$$

Proof: The inclusion $R_{\mathfrak{p}}[x] \subseteq (R[x])_{\mathfrak{p}R[x]} \cap F[x]$ is obvious. Now, let $f(x)$ be an element of $F[x]$. Then $f(x)$ can be written in the form $r(x)/\xi$, where $r(x) \in R[x]$ and $\xi \in R$. If, moreover, $f(x)$ belongs to $(R[x])_{\mathfrak{p}R[x]}$, then there exist $s(x), t(x) \in R[x]$ with $t(x) \notin \mathfrak{p}R[x]$ such that $f(x) = s(x)/t(x)$. Thus, we have

$$f(x) = \frac{r(x)}{\xi} = \frac{s(x)}{t(x)}.$$

All the coefficients of the product $r(x)t(x)$ belong to ξR. Due to Lemma F, if $r(x) = \sum_i a_i x^i$ and $t(x) = \sum_j b_j x^j$, then all the products $a_i b_j$ belong to ξR. Now, the fact that $t(x) \notin \mathfrak{p}R[x]$ implies there exists j_0 such that $b_{j_0} \notin \mathfrak{p}$. Since $a_i b_{j_0} \in \xi R$, for all i, we deduce that $b_{j_0} f(x) = (b_{j_0} r(x))/\xi \in R[x]$ and so all the coefficients of $f(x)$ belong to $R_{\mathfrak{p}}$.

This result generalises to polynomial rings and Laurent polynomial rings in multiple variables.

Lemma B (C.)

Let R be an integrally closed domain and let F be its field of fractions. Let $s(x)$ and $t(x)$ be two elements of $R[x]$ such that $s(x)/t(x) \in F[x]$. If one of the coefficients of $t(x)$ is a unit in R, then $s(x)/t(x) \in R[x]$.

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Proof: Since $s(x)/t(x) \in F[x]$, there exist $r(x) \in R[x]$ and $\xi \in R$ such that

$$\frac{s(x)}{t(x)} = \frac{r(x)}{\xi}.$$

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Preview

Let $\lambda, \mu \in \mathbb{C}^\times$ and $a, b, c \in \mathbb{N}$ with $\gcd(a, b, c) = 1$. The polynomial $\lambda\, x^a y^b + \mu\, z^c$ is irreducible in $\mathbb{C}[x, y, z]$, because $\lambda x + \mu$ is irreducible in $\mathbb{C}[x]$.