Two fundamental conjectures on the structure of Hecke algebras

Part II: The BMM

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(joint work with C. Boura, E. Chavli & K. Karvounis)

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It is known to hold for:

- the real reflection groups by Bourbaki;
- the groups G_4, G_{12}, G_{22}, G_{24} by Malle–Michel (G_4 also by Marin–Wagner).
- *the infinite series $G(de, e, r)$ by Bremke–Malle & Malle–Mathas (?).*

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STEP 3: Check that the extra third condition holds.

The C++ algorithm

For any $b_i, b_j \in \mathcal{B}_n$, our C++ program expresses $b_i b_j$ as a linear combination of the elements of $\mathcal{B}_n$. Then $\tau(b_i b_j)$ is the coefficient of 1 in this linear combination.

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The inputs of the algorithm are the following:

I1. The basis $\mathcal{B}_n$.

I2. The generating relations of the Hecke algebra $\mathcal{H}(G_n)$.

I3. The "special cases": these are some equalities computed by hand which express a given element of $\mathcal{H}(G_n)$ as a sum of other elements in $\mathcal{H}(G_n)$.

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The case of G_4

We have $$\mathcal{B}_4 = \left\{ \begin{array}{l} 1, s, s^2, t^2, t, t^2s, ts, t^2s^2, ts^2, st^2, st, st^2s, sts, st^2s^2, sts^2, \\ s^2t^2, s^2t, s^2t^2s, s^2ts, s^2t^2s^2, s^2ts^2, ststst, ststst s, ststst s^2 \end{array} \right\}.$$

Running the C++ program takes about 1 hour on an Intel Core i5 CPU.

The SAGE algorithm

Our SAGE program produces the matrix A row by row, using the distinctive pattern of the basis $\mathcal{B}_n$: there exists a set $\mathcal{E}_n$ with $1 \in \mathcal{E}_n$ such that

$$\mathcal{B}_n = \{z^k e \mid e \in \mathcal{E}_n,\ k = 0, 1, \ldots, |Z(G_n)| - 1\},$$

where $z \in Z(\mathcal{H}(G_n))$ is the image of a generator of the centre of the corresponding braid group.

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We have $\mathcal{E}_n \leftrightarrow G_n/Z(G_n) \cong \begin{cases} \mathfrak{A}_4 & \text{for } n = 5, 6, 7; \\ \mathfrak{S}_4 & \text{for } n = 8, 13. \end{cases}$

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The inputs of the SAGE algorithm are the coefficients of the following elements when written as linear combinations of the elements of $\mathcal{B}_n$:

I1. gb_j for all $b_j \in \mathcal{B}_n$, where g runs over the generators of $\mathcal{H}(G_n)$.

I2. $z^{|Z(G_n)|} = z \cdot z^{|Z(G_n)|-1}$.

The case of G_5

- $\mathcal{H}(G_5) = \left\langle s,\ t \ \middle|\ stst = tsts,\ \ s^3 = as^2 + bs + c,\ \ t^3 = dt^2 + et + f \right\rangle.$

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b_{12k+2} &= b_{12k+1} \cdot s, & b_{12k+8} &= b_{12k+6} \cdot t,\\
b_{12k+3} &= b_{12k+2} \cdot s, & b_{12k+9} &= b_{12k+7} \cdot t,\\
b_{12k+4} &= b_{12k+1} \cdot t, & b_{12k+10} &= f^{-1}(b_{12k+5} - db_{12k+4} - eb_{12k+1}) \cdot s,\\
b_{12k+5} &= b_{12k+4} \cdot t, & b_{12k+11} &= b_{12k+10} \cdot t,\\
b_{12k+6} &= b_{12k+2} \cdot t, & b_{12k+12} &= b_{12k+11} \cdot t.\\
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Let $sb_j = \sum_\ell \lambda^s_{j,\ell} b_\ell$, $tb_j = \sum_\ell \lambda^t_{j,\ell} b_\ell$ and $z^6 = \sum_\ell \mu_\ell b_\ell$.

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- $\tau(b_{12k+10} b_j) = f^{-1} \sum_\ell \lambda^s_{j,\ell} (\tau(b_{12k+5} b_\ell) - d\tau(b_{12k+4} b_\ell) - e\tau(b_{12k+1} b_\ell))$.

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Our program worked for G_4 and G_6. It produced A for G_8, but could not calculate $\det(A)$. It could not even establish STEP 2 for G_5 and G_7.

The extra condition

Malle and Michel have shown that, since

1. each element of $\mathcal{B}_n$ corresponds to a distinct element of G_n,
2. $\tau(b) = \delta_{1b}$ for all $b \in \mathcal{B}_n$, and
3. $\mathcal{B}_n$ is a basis of $\mathcal{H}(G_n)$ as an R_{G_n}-module,

the extra condition of the BMM symmetrising trace conjecture translates as:

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- We used GAP to prove Formula (3) for G_4, G_6 and G_8.
- We directly proved Formula (2) for G_5, G_7 and G_{13}, by expressing $\tau\left(z^{|Z(G_n)|}b^{-1}\right)$ as a linear combination of entries of the matrix A.

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Our C++ program has expressed gb_j as a linear combination of the elements of $\mathcal{B}_n$, for every generator g of $\mathcal{H}(G_n)$ and every $b_j \in \mathcal{B}_n$. This in fact allows us to express any product of the generators, and thus any element, of $\mathcal{H}(G_n)$ as a linear combination of the elements of $\mathcal{B}_n$.