Three generalisations of the Temperley-Lieb algebra

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Joint work with Guillaume Pouchin

The Iwahori-Hecke algebra $\mathcal{H}_n(q)$ and the Temperley-Lieb algebra $TL_n(q)$

$$\mathcal{H}_n(q) = \left\langle g_1, \dots, g_{n-1} \;\middle|\; \begin{array}{l} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \\ g_i g_j = g_j g_i \quad \text{si} \quad |i-j| > 1 \\ g_i^2 = q + (q-1) g_i \end{array} \right\rangle \qquad \mathbb{C}(q)\text{-algebra}$$

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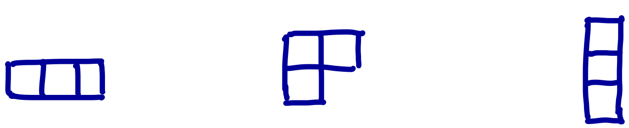
Example : $n = 3$

(3) $\qquad$ $(2,1)$ $\qquad$ $(1,1,1)$ $\qquad \longleftarrow \lambda$



$\longleftarrow Y(\lambda)$

3 $\qquad$ 2 $\qquad$ 1 $\qquad \longleftarrow c(\lambda) = \#$ columns of $Y(\lambda)$

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$n \geqslant 3 \quad G_i := 1 + g_i + g_{i+1} + g_i g_{i+1} + g_{i+1} g_i + g_i g_{i+1} g_i$

$$TL_n(q) := \mathcal{H}_n(q) / I \quad , \qquad I = \langle G_i \mid 1 \leqslant i \leqslant n-2 \rangle$$

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Branching rule for $\mathfrak{S}_n$

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \rho^{\lambda} = \bigoplus \rho^{Y(\lambda) \setminus \square}$$

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POLYNOMIAL KNOT INVARIANTS

Markov trace on $TL_n(q) \rightsquigarrow$ Jones polynomial [1984]

$\mathcal{H}_n(q) \rightsquigarrow$ HOMFLY-PT polynomial [1985]

The Yokonuma-Hecke algebra of type A

Let $d \in \mathbb{Z}_{>0}$.

$$Y_{d,n}(q) = \left\langle \begin{matrix} g_1, \dots, g_{n-1} \\ t_1, \dots, t_n \end{matrix} \;\middle|\; \begin{matrix} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \quad \text{if } |i-j| > 1 \\ t_j^d = 1 \quad , \; t_i t_j = t_j t_i \quad , \quad t_j g_i = g_i t_{s_i(j)} \\ g_i^2 = (q-1) e_i g_i + q \end{matrix} \right\rangle$$

where $e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}$ is an idempotent. $\mathbb{C}(q)$-algebra

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$$\mathrm{Irr}(Y_{d,n}(q)) \longleftrightarrow \mathrm{Irr}(\underbrace{\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n}_{\overset{\shortparallel}{G(d,1,n)}}) \longleftrightarrow \{ d\text{-partitions of } n \} =: \mathcal{P}_d(n)$$

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Markov trace on $Y_{d,n}(q) \rightsquigarrow \Theta$-invariant stronger than HOMFLYPT

[Juyumaya-Lambropoulou '13 , C.-Juyumaya-Karvounis-Lambropoulou '20]

The Yokonuma-Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou, 2014]

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Example: $d = 2$ $\quad$ $G(2,1,n) = B_n$

$$\mathrm{Res}^{B_n}_{\mathfrak{S}_n}(\rho^{\lambda}) = \sum \underbrace{c^{\mu}_{\lambda^1,\lambda^2}}_{\text{Littlewood-Richardson coefficients}} \mu$$

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Littlewood-Richardson coefficients

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<u>Example</u>: $\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \square \right)$ ✓ $\qquad (\square\square, \square)$ ✗ $\qquad (\emptyset, \square\square\square)$ ✗

The Yokonuma-Temperley-Lieb algebra

[Goundaroulis - Juyumaya - Kontogeorgis - Lambropoulou, 2014]

$YTL_{d,n}(q) := Y_{d,n}(q) / \langle G_1 \rangle \qquad G_1 = 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1$

$\mathrm{Irr}(YTL_{d,n}(q)) \longleftrightarrow \{ \lambda \in \mathcal{P}_d(n) \mid \mathrm{Res}^{G(d,1,n)}_{\mathfrak{S}_n} \rho^\lambda \left(\sum_{w \in \mathfrak{S}_3} w \right) = 0 \}$

<u>Example</u>: $d = 2 \quad G(2,1,n) = B_n$

$\mathrm{Res}^{B_n}_{\mathfrak{S}_n} (\rho^\lambda) = \sum c^{\mu}_{\lambda^1, \lambda^2} \, \mu$

↑ Littlewood - Richardson coefficients

<u>Theorem</u> [C.-Pouchin '15] $\mathrm{Irr}(YTL_{d,n}(q)) \longleftrightarrow \{ \lambda \in \mathcal{P}_d(n) \mid \sum_{i=1}^{d} c(\lambda^i) \leq 2 \}$

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Framisation of the Temperley-Lieb algebra

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Complex Reflection Temperley-Lieb algebra

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\begin{tikzcd}
Y_{d,n}(q) \arrow[r, two heads] \arrow[d, "t_j \mapsto 1"'] & CTL_{d,n}(q) \arrow[r, two heads] \arrow[d, two heads] & FTL_{d,n}(q) \arrow[r, two heads] \arrow[dl, two heads] & YTL_{d,n}(q) \arrow[dll, two heads] \\
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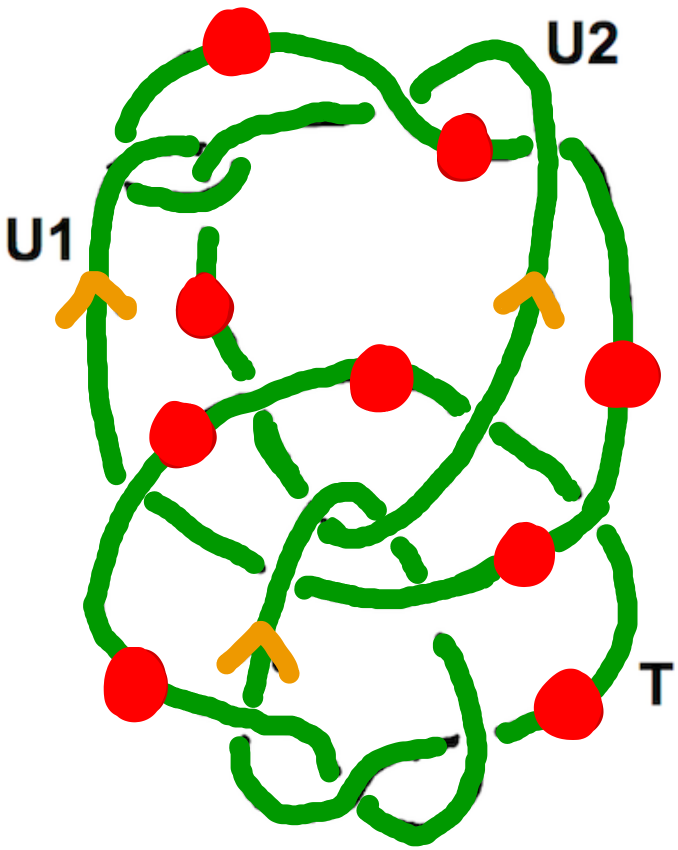
The link LLL(0)

has the same Jones polynomial as the disjoint union of 3 unkots ($1 \in B_3$). However, the θ-invariant distinguishes the two [C., 2019]



Merry Christmas!