Representation Theory of Hecke algebras & connections with Cherednik algebras

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Iwahori-Hecke algebras

Let (W, S) be a finite Coxeter system,

$$W = \langle\ S \mid s^2 = 1,\ \underbrace{ststst\ldots}_{m_{st}} = \underbrace{tststs\ldots}_{m_{st}}\ \ \forall s, t \in S\ \rangle.$$

$$\mathfrak{S}_n = \left\langle\, s_1, s_2, \ldots, s_{n-1} \;\middle|\; s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i\ \text{ if } |i-j| > 1 \,\right\rangle$$

Let $L : S \to \mathbb{Z}$ be a function such that $L(s) = L(t)$ whenever s and t are conjugate in W. Let q be an indeterminate.

$$\mathcal{H}_q(W) = \langle\, (T_s)_{s\in S} \mid \underbrace{T_s T_t T_s \ldots}_{m_{st}} = \underbrace{T_t T_s T_t \ldots}_{m_{st}},\ (T_s - q^{L(s)})(T_s + q^{-L(s)}) = 0\ \forall s, t\,\rangle.$$

$$\mathcal{H}_q(\mathfrak{S}_n) = \left\langle\ T_1, T_2, \ldots, T_{n-1} \;\middle|\; \begin{array}{l} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i \ \text{ if } |i-j| > 1, \\ (T_i - q^K)(T_i + q^{-K}) = 0 \end{array} \right\rangle$$

where $K := L(s_1) = \cdots = L(s_{n-1})$.

The a-function

Let $w \in W$. Let $w = s_{i_1} s_{i_2} \dots s_{i_r}$ be a *reduced expression* for w. Set $T_w := T_{s_{i_1}} T_{s_{i_2}} \dots T_{s_{i_r}}$. The algebra $\mathcal{H}_q(W)$ is a free $\mathbb{C}[q, q^{-1}]$-module with basis $(T_w)_{w \in W}$.

Let $\tau : \mathcal{H}_q(W) \to \mathbb{C}[q, q^{-1}]$ be the linear map defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ if $w \neq 1$. The map τ is a *symmetrising trace*. By extension of scalars to $\mathbb{C}(q)$, we have

$$\tau = \sum_{V \in \mathrm{Irr}(\mathbb{C}(q)\mathcal{H}_q(W))} \frac{1}{s_V} \chi_V$$

for some $s_V \in \mathbb{C}[q, q^{-1}]$ (Schur elements).

The algebra $\mathbb{C}(q)\mathcal{H}_q(W)$ is (split) semisimple, hence

$$\begin{array}{ccc} \mathrm{Irr}(W) & \leftrightarrow & \mathrm{Irr}(\mathbb{C}(q)\mathcal{H}_q(W)) \\ E & \mapsto & V_E \end{array} .$$

Let $E \in \mathrm{Irr}(W)$. We set

$$a(E) := -\mathrm{valuation}(s_{V_E}) \quad \text{and} \quad A(E) := -\mathrm{degree}(s_{V_E}).$$

Canonical basic sets

Let $\theta : \mathbb{C}[q, q^{-1}] \to \mathbb{C},\ q \mapsto \xi$ be a ring homomorphism such that $\xi \in \mathbb{C}^\times$. The algebra $\mathbb{C}\mathcal{H}_\xi(W)$ is not necessarily semisimple. We obtain a decomposition matrix

$$D_\xi = ([V_E : M])_{E \in \mathrm{Irr}(W), M \in \mathrm{Irr}(\mathbb{C}\mathcal{H}_\xi(W))}.$$

A canonical basic set for $\mathcal{H}_\xi(W)$ is a subset $\mathcal{B}_\xi$ of $\mathrm{Irr}(W)$ such that

1. $\mathrm{Irr}(\mathbb{C}\mathcal{H}_\xi(W)) \leftrightarrow \mathcal{B}_\xi, \quad M \mapsto E^M$;
2. $[V_{E^M} : M] = 1$ for all $M \in \mathrm{Irr}(\mathbb{C}\mathcal{H}_\xi(W))$;
3. if $[V_E : M] \neq 0$ for some $E \in \mathrm{Irr}(W)$, then either $E^M = E$ or $a(E^M) < a(E)$.

$$D_\xi = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}}_{\mathrm{Irr}(\mathbb{C}\mathcal{H}_\xi(W))} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \mathcal{B}_\xi \\ \\ \\ \\ \end{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} \mathrm{Irr}(W)$$

Theorem [Geck, Rouquier, Jacon, C.]

Canonical basic sets exist for finite Coxeter groups.

It is well-known that

$$\mathrm{Irr}(\mathfrak{S}_n) \leftrightarrow \left\{ \lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 1) : \sum_{i=1}^{r} \lambda_i = n \right\}.$$

Let ξ be a primitive root of unity of order $e > 1$. Then we have

$$\mathcal{B}_\xi = \{\lambda \mid \lambda \text{ is } e\text{-regular}\} \qquad \text{if } K > 0$$

and

$$\mathcal{B}_\xi = \{\lambda \mid \lambda \text{ is } e\text{-restricted}\} \quad \text{if } K < 0$$

where $K := L(s_1) = \cdots = L(s_{n-1})$.

Cellular structure

Under Lusztig's conjectures (P1)–(P15), Iwahori-Hecke algebras are **cellular** :

- Cell modules $M(E)_{E \in \mathrm{Irr}(W)}$.
- Symmetric bilinear form $\langle \ , \ \rangle$ on cell modules.

Theorem [Graham-Lehrer]

Set $D(E) := M(E)/\mathrm{rad}_{\langle \ , \ \rangle} M(E)$. We have that

1. $D(E)$ is either 0 or a simple $\mathbb{C}\mathcal{H}_\xi(W)$-module.
2. $\{D(E) \,|\, D(E) \neq 0\} = \mathrm{Irr}(\mathbb{C}\mathcal{H}_\xi(W))$.

We have

$$\{D(E) \,|\, D(E) \neq 0\} = \{D(E) \,|\, E \in \mathcal{B}_\xi\}.$$

Rational Cherednik algebras

Let $\mathfrak{h}$ be the reflection representation of W, and let $V = \mathfrak{h} \oplus \mathfrak{h}^*$. We denote by $\mathbf{S}$ the set of all reflections in W. For $\mathbf{s} \in \mathbf{S}$, take

- $\alpha_{\mathbf{s}} \in \mathfrak{h}^*$: basis of $\mathrm{Im}(\mathbf{s} - \mathrm{id}_V)|_{\mathfrak{h}^*}$.
- $\alpha_{\mathbf{s}}^\vee \in \mathfrak{h}$: basis of $\mathrm{Im}(\mathbf{s} - \mathrm{id}_V)|_{\mathfrak{h}}$.

Let $\mathbf{c} : \mathbf{S} \to \mathbb{C}$ be a function such that $\mathbf{c}(\mathbf{s}) = \mathbf{c}(\mathbf{t})$ whenever $\mathbf{s}$ and $\mathbf{t}$ are conjugate in W.

$$\mathrm{H}_{\mathbf{c}}(W) = \frac{TV^* \rtimes W}{[x, x'] = 0,\ [y, y'] = 0,\ [y, x] = x(y) - \sum_{\mathbf{s} \in \mathbf{S}} \mathbf{c}(\mathbf{s})\, \alpha_{\mathbf{s}}(y)\, x(\alpha_{\mathbf{s}}^\vee)\, \mathbf{s}}$$

for $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$.

$W = \mathfrak{S}_n$, $\mathbf{S} = \{s_{ij} = (i\,j)\}$, $(x_1, \ldots, x_n)$ basis of $\mathfrak{h}^*$, $(y_1, \ldots, y_n)$ basis of $\mathfrak{h}$.
$\sigma \cdot x_i = x_{\sigma(i)} \cdot \sigma,\ \sigma \cdot y_i = y_{\sigma(i)} \cdot \sigma,\ \forall \sigma \in \mathfrak{S}_n$.
Take $\alpha_{ij} := x_i - x_j$, $\alpha_{ij}^\vee := y_i - y_j$, for $1 \le i < j \le n$, and $\mathbf{c} \in \mathbb{C}$.
Then the commutation relations in $\mathrm{H}_{\mathbf{c}}(W)$ are:
$[x_i, x_j] = 0,\ [y_i, y_j] = 0,\ [y_i, x_i] = 1 - \mathbf{c} \sum_{j \neq i} s_{ij}$ and $[y_i, x_j] = \mathbf{c} s_{ij}$ for $i \neq j$.

The category $\mathcal{O}$

$\mathcal{O}$ = the category of finitely generated $\mathrm{H}_{\mathbf{c}}(W)$-modules locally nilpotent for the action of $\mathfrak{h}$.

- Standard modules $\Delta(E) = \mathrm{Ind}_{\mathbb{C}[\mathfrak{h}]\rtimes W}^{\mathrm{H}_{\mathbf{c}}(W)}(E), \quad E \in \mathrm{Irr}(W)$.
- Simple modules $L(E) = \mathrm{Head}(\Delta(E)), \quad E \in \mathrm{Irr}(W)$.
- Decomposition matrix $D^{\mathcal{O}} = ([\Delta(F) : L(E)])_{E,F\in\mathrm{Irr}(W)}$.
- $[\Delta(E) : L(E)] = 1$, for all $E \in \mathrm{Irr}(W)$.
- There exists an ordering $<$ on the standard modules (and consequently on $\mathrm{Irr}(W)$) such that if $[\Delta(F) : L(E)] \neq 0$, then either $E = F$ or $E < F$.

A famous ordering on the category $\mathcal{O}$ is the following:

$$E < F \text{ if and only if } c(F) - c(E) \in \mathbb{Z}_{>0}$$

where $c(E)$ is the scalar with which the **Euler element** $\in Z(\mathbb{C}W)$ acts on E.

The KZ functor

There exists an exact functor

$$\mathrm{KZ} : \mathcal{O} \longrightarrow \mathcal{H}_\xi(W) - \mathrm{mod}$$

where $\xi = \exp(2\pi i \mathbf{c}(s)/L(s))$. We have:

1. $\mathrm{KZ}(L(E))$ is either 0 or a simple $\mathcal{H}_\xi(W)$-module.
2. $\{\mathrm{KZ}(L(E)) \,|\, \mathrm{KZ}(L(E)) \neq 0\} = \mathrm{Irr}(\mathbb{C}\mathcal{H}_\xi(W))$.
3. If $\mathrm{KZ}(L(E)) \neq 0$, then $[\Delta(F) : L(E)] = [\mathrm{KZ}(\Delta(F)) : \mathrm{KZ}(L(E))]$. Thus,
   - $[\mathrm{KZ}(\Delta(E)) : \mathrm{KZ}(L(E))] = 1$;
   - If $[\mathrm{KZ}(\Delta(F)) : \mathrm{KZ}(L(E))] \neq 0$, then either $E = F$ or $E < F$.

Proposition [C-Gordon-Griffeth]

For all $E \in \mathrm{Irr}(W)$, we have

$$c(E) = a(E) + A(E).$$

Main result

Theorem [C-Gordon-Griffeth]

The a-function is an ordering on the category $\mathcal{O}$. This in turn implies that

1. there exists a canonical basic set $\mathcal{B}_\xi$ for $\mathcal{H}_\xi(W)$;
2. $\mathrm{KZ}(L(E)) \neq 0$ if and only if $E \in \mathcal{B}_\xi$.

Moreover, we have $\mathrm{KZ}(\Delta(E)) \cong M(E)$ (cell module) for all $E \in \mathrm{Irr}(W)$.

Remark: The above theorem holds for the complex reflection group $G(\ell, 1, n) \cong (\mathbb{Z}/\ell\mathbb{Z})^n \rtimes \mathfrak{S}_n$ (without the use of Ariki's theorem).

The case of $G(\ell, 1, n)$

Let $e \in \mathbb{Z}_{>0}$ and $(s_0, s_1, \ldots, s_{\ell-1}) \in \mathbb{Z}^\ell$. We consider the specialised Ariki-Koike algebra defined by

- generators : $T_0, T_1, \ldots, T_{n-1}$
- relations :

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$$(T_0 - \zeta_e^{s_0})(T_0 - \zeta_e^{s_1}) \cdots (T_0 - \zeta_e^{s_{\ell-1}}) = 0$$

$$(T_i - \zeta_e)(T_i + 1) = 0, \quad \text{for all } i = 1, \ldots, n-1.$$

We set $m_j := \ell s_j - je$ and $m := (m_j)_{0 \leq j \leq \ell-1}$. We consider the cyclotomic Ariki-Koike algebra $\mathcal{H}_{q,m}$ with relations:

$$\left.\begin{aligned} &(T_0 - q^{m_0})(T_0 - \zeta_\ell q^{m_1}) \cdots (T_0 - \zeta_\ell^{\ell-1} q^{m_{\ell-1}}) = 0 \\ &(T_i - q^\ell)(T_i + 1) = 0, \quad \text{for all } i = 1, \ldots, n-1 \end{aligned}\right\} \quad \theta : q \mapsto \zeta_{e\ell}.$$

$$\Pi_n^\ell = \{\ell\text{-partitions of } n\} \quad \leftrightarrow \quad \mathrm{Irr}(G(\ell,1,n)) \quad \leftrightarrow \quad \mathrm{Irr}(\mathcal{H}_{q,m})$$

$$\lambda = (\lambda^{(0)}, \ldots, \lambda^{(\ell-1)}) \quad \mapsto \quad E^\lambda$$

Theorem [Geck-Jacon]

Canonical basic set $\leftrightarrow$ {Uglov ℓ-partitions}.

$$[\lambda] := \{\gamma = (a,b,c) \mid 0 \le c \le \ell - 1,\ a \ge 1,\ 1 \le b \le \lambda_a^{(c)}\}.$$

$$\mathrm{cont}(\gamma) := b(\gamma) - a(\gamma) \quad \text{and} \quad \vartheta(\gamma) := \mathrm{cont}(\gamma) + s_c.$$

Theorem [Dunkl-Griffeth]

Let λ, $\lambda' \in \Pi_n^\ell$. If $[\Delta(E^\lambda) : L(E^{\lambda'})] \neq 0$, then there exist orderings on the nodes $\gamma_1, \gamma_2, \ldots, \gamma_n$ and $\gamma'_1, \gamma'_2, \ldots, \gamma'_n$ of λ and λ' respectively, and integers $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{Z}_{\ge 0}$ such that, for all $1 \le i \le n$,

$$\mu_i \equiv c(\gamma_i) - c(\gamma'_i) \bmod \ell \quad \text{and} \quad \mu_i = c(\gamma_i) - c(\gamma'_i) + \frac{\ell}{e}(\vartheta(\gamma'_i) - \vartheta(\gamma_i)).$$

Theorem [C-Gordon-Griffeth]

Let λ, $\lambda' \in \Pi_n^\ell$. If $[\Delta(E^\lambda) : L(E^{\lambda'})] \neq 0$, then either $\lambda' = \lambda$ or $a(E^{\lambda'}) < a(E^\lambda)$.

Proof: Let γ and γ' be nodes of ℓ-partitions. We write $\gamma \prec \gamma'$ if we have

$$\vartheta(\gamma) < \vartheta(\gamma') \quad \text{or} \quad \vartheta(\gamma) = \vartheta(\gamma') \text{ and } c(\gamma) > c(\gamma').$$

Using the result by Dunkl and Griffeth, we can order the nodes $\gamma_1, \gamma_2, \ldots, \gamma_n$ et $\gamma'_1, \gamma'_2, \ldots, \gamma'_n$ of λ and λ' respectively such that, for all $1 \leq i \leq n$,

$$\gamma_i \prec \gamma'_i \quad \text{or} \quad \gamma_i = \gamma'_i.$$

Then we can prove that either $\lambda' = \lambda$ or $a(E^{\lambda'}) < a(E^\lambda)$.