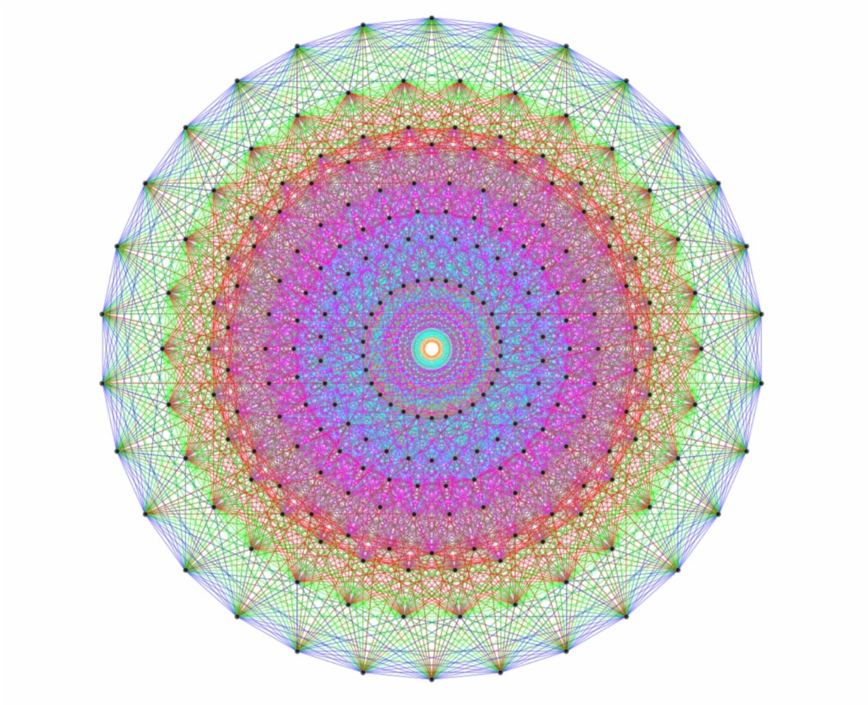
The Symmetric Group
Mysteries & Miracles



Maria Chlouveraki

National and Kapodistrian
University of Athens

GWM

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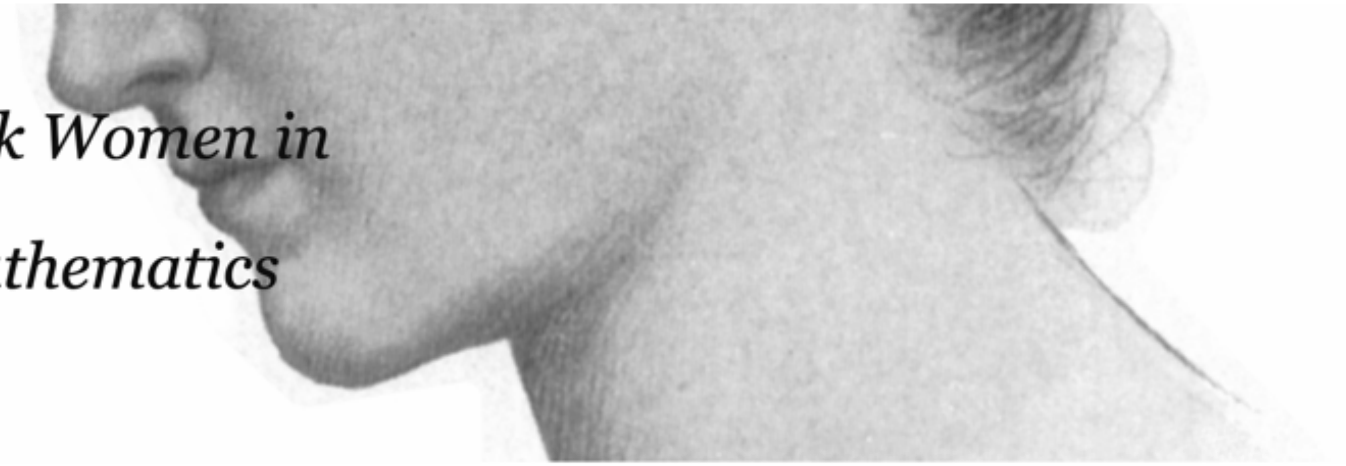

Greek Women in Mathematics

Ελληνικά

Welcome to our page!

Who we are: Greek Women in Mathematics is an association of women working in mathematics in Greece and abroad. Anyone who is enthusiastic and passionate about mathematics, from high school level to full professorship but also outside academia, is welcome to join!

Why Greek Women in Mathematics: The aim of this association is to encourage and support women that want to study and work in the field of mathematics by creating a network. Another important task is giving visibility to female mathematicians and offering opportunities for scientific collaborations.

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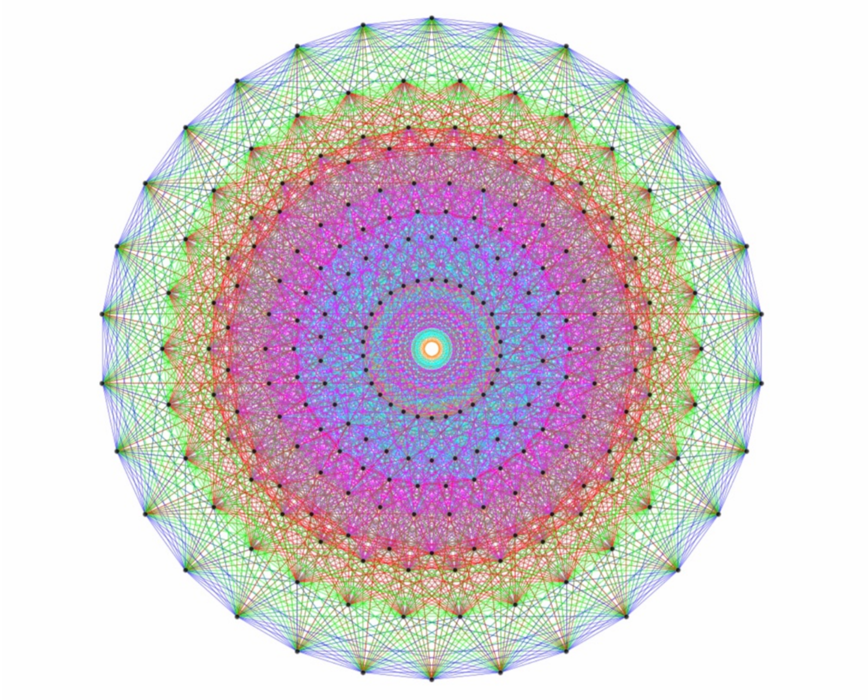

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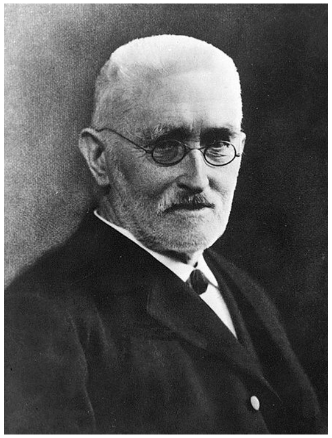
One letter, two mathematicians



Richard Dedekind

(1831 - 1916)

Ferdinand G. Frobenius

(1849 - 1917)

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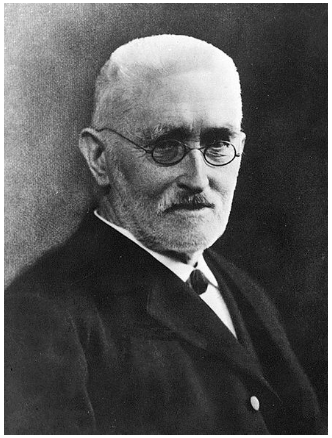

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" Take the multiplication table of a finite group G, and turn it into a matrix X_G by replacing every entry g of this table by a variable x_g. Then the determinant of X_G factors into a product of irreducible polynomials in $\{x_g\}$, each of which occurs with multiplicity equal to its degree."

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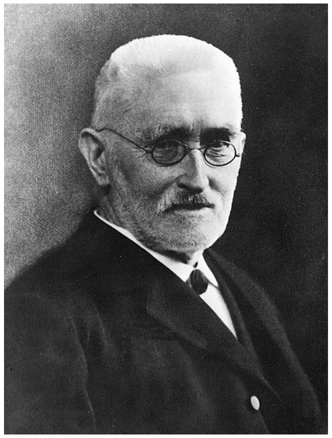

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Ferdinand G. Frobenius
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"In April of this year, Dedekind gave me an exercise ... [whose] solution, which I hope to be able to present soon, led me to the generalisation of the notion of a character to arbitrary finite groups. I want to develop this notion here, since I believe that by its introduction, group theory should undergo a major advancement and enrichment".

Representation Theory

G finite group, k algebraically closed field, V k-vector space

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(V, ρ) REPRESENTATION OF G

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(V, ρ) REPRESENTATION OF G

$\chi(g) := \mathrm{Trace}(\rho(g))$ CHARACTER OF (V,ρ)

$\chi : G \longrightarrow k$

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In characteristic 0, every representation is a direct sum of irreducible ones.

Rk: $kG = \{ \sum_{g \in G} a_g g \mid a_g \in k \}$, V is a kG-module

The symmetric group $\mathfrak{S}_n$

$\mathfrak{S}_n = \langle (1,2), (2,3), \ldots, (n-1,n) \rangle = \langle s_1, s_2, \ldots, s_{n-1} \rangle$ where $s_i = (i, i+1)$

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$$\mathfrak{S}_3 = \langle \underset{s_1}{\underset{\shortparallel}{(1,2)}}, \underset{s_2}{\underset{\shortparallel}{(2,3)}} \rangle$$

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trivial

$\dim V = 1$

$s_1 \longmapsto 1$

$s_2 \longmapsto 1$

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trivial	sign
$\dim V = 1$	$\dim V = 1$
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trivial	sign	reflection
$\dim V = 1$	$\dim V = 1$	$\dim V = 2$, $V = \langle (1,-1,0),(0,1,-1) \rangle \subseteq \mathbb{C}^3$
$s_1 \longmapsto 1$	$s_1 \longmapsto -1$	$s_1 \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, $s_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$
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$$\mathbb{C}^3 = V \oplus \langle (1,1,1)\rangle$$

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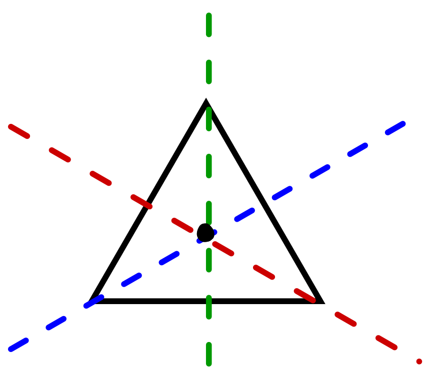

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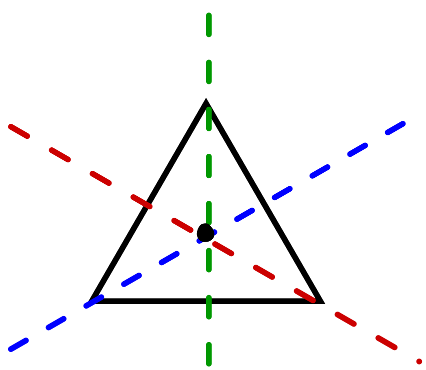

$$s_1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad s_2 \mapsto \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

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$$\sum_{\chi \in \mathrm{Irr}(\mathbb{C}G)} \chi(1)^2 = |G|$$

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$$\mathrm{Irr}(\mathbb{C}\mathfrak{S}_n) \longleftrightarrow \{\text{partitions of } n\}$$

$$\lambda = (\lambda_1, \dots, \lambda_r) \quad \text{with } \lambda_1 \geq \dots \geq \lambda_r \geq 1$$

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(3)	$(1,1,1)$	$(2,1)$

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Modular representation theory

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kG is semisimple $\iff$ $p \nmid |G|$

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(1901-1977)

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Decomposition matrix

$$\underbrace{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \\ & & \end{pmatrix}}_{\mathrm{Irr}(kG)} \left.\vphantom{\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}}\right\} \mathrm{Irr}(\mathbb{C}G)$$

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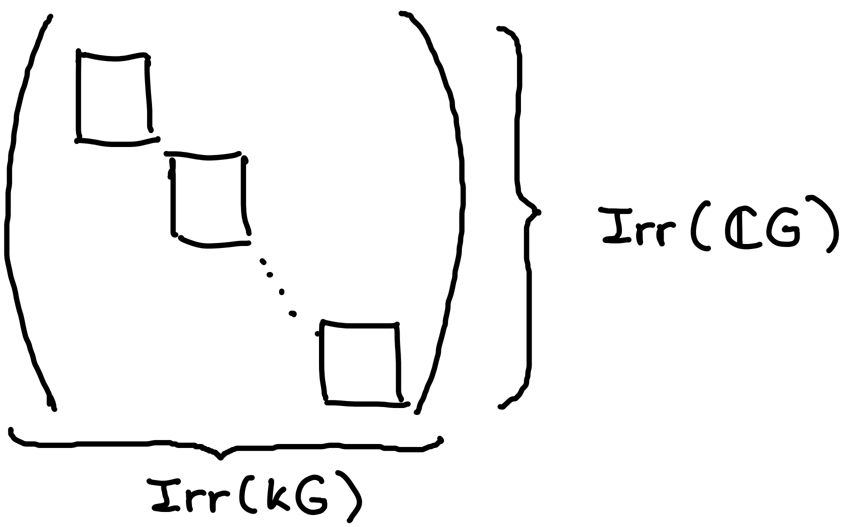

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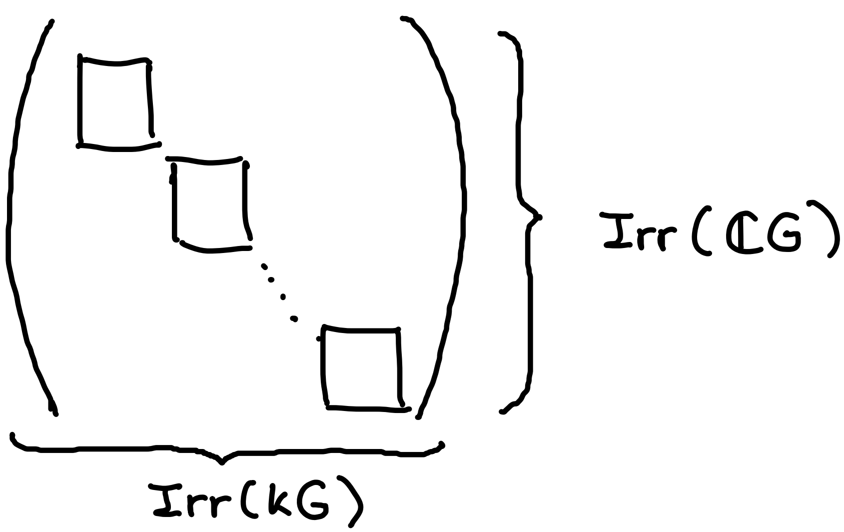

$\chi \in \operatorname{Irr}(\mathbb{C}G)$ alone in its block

$\Updownarrow$

$p \nmid \dfrac{|G|}{\chi(1)}$

Schur element of χ

Modular representation theory

$\operatorname{char} k = p > 0$

kG is semisimple $\Leftrightarrow$ $p \nmid |G|$

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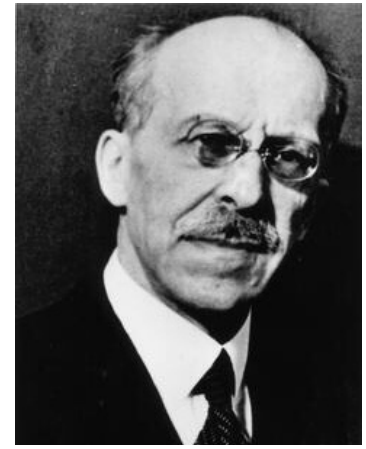
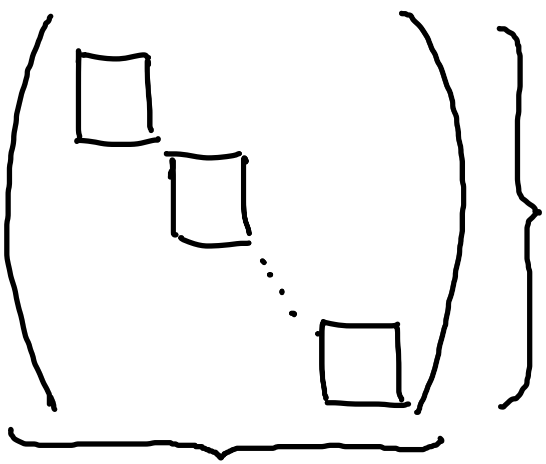


Issai Schur

(1875-1941)

Decomposition matrix



$\mathrm{Irr}(\mathbb{C}G)$

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The decomposition matrix of $\mathfrak{S}_n$

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IS NOT KNOWN !!!

The decomposition matrix of $\mathfrak{S}_n$

$$D_p = \underbrace{\left(\begin{array}{ccccc} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & * & & \ddots & \\ & & & & 1 \\ \hline * & * & \cdots & & * \end{array}\right)}_{\mathrm{Irr}(k\mathfrak{S}_n)} \Bigg\} \, \mathrm{Irr}(\mathbb{C}\mathfrak{S}_n)$$

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$$\{\, p\text{-regular partitions of } n \,\}$$

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<u>Example</u>: $n = 3$ $\quad |\mathfrak{S}_3| = 6$ $\quad p \in \{2, 3\}$ $\quad (1,1,1)$ is not p-regular

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$p = 2$

$$D_2 = \begin{array}{cc} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{array}\right) & \begin{array}{l} (3) \\ (2,1) \\ (1,1,1) \end{array} \\ \begin{array}{cc} (3) & (2,1) \end{array} & \end{array}$$

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$$\text{Schur elt} \qquad 6 \qquad 3$$

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Schur elt $\quad 6 \quad\quad 3$

$p = 3$

$$\begin{array}{cc} D_3 = \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{array}\right) & \begin{array}{l} (3) \\ (2,1) \\ (1,1,1) \end{array} \\ \begin{array}{cc} \quad\quad\quad (3) & (2,1) \end{array} & \end{array}$$

The Iwahori-Hecke algebra of $\mathfrak{S}_n$

$$\mathfrak{S}_n = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \\ s_i s_j = s_j s_i & \text{if } |i-j| > 1 \\ s_i^2 = 1 & \end{array} \right\rangle$$

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$$\mathcal{H}_q(\mathfrak{S}_n) = \left\langle T_1, \dots, T_{n-1} \;\middle|\; \begin{array}{ll} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \\ T_i T_j = T_j T_i & \text{if } |i-j| > 1 \\ T_i^2 = (q-1) T_i + q & \end{array} \right\rangle$$

$\mathbb{Z}[q, q^{-1}]$-algebra

The Iwahori-Hecke algebra of $\mathfrak{S}_n$

$$\mathfrak{S}_n = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \\ s_i s_j = s_j s_i & \text{if } |i-j|>1 \\ s_i^2 = 1 & \end{array} \right\rangle$$

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$\mathbb{C}(q) \otimes \mathcal{H}_q(\mathfrak{S}_n)$ is semisimple

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James's conjecture

$$D_n = \left.\underbrace{\left(\begin{array}{ccccc} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ * & & & \ddots & \\ & & & & 1 \\ \hline * & * & & \cdots & * \end{array}\right)}_{\mathrm{Irr}(\mathbb{C}\mathcal{H}_\eta(\mathfrak{S}_n))}\right\} \mathrm{Irr}(\mathbb{C}(q)\mathcal{H}_q(\mathfrak{S}_n))$$

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<u>Counter-example</u> [Williamson, 2017] $n = 1.744.860$

Interlude

There are "philosophical" questions about the value and status of a conjecture. If a conjecture turns out to be false, it does not necessarily mean that work in that direction should cease. If that "first approximation" to the truth turns out not to be completely accurate, surely that is a spur to attempt a better approximation to the truth. This is normal, and arguably inevitable, in the evolution of scientific theories. Once a mathematical theorem is genuinely proven true (within an agreed axiomatic system) it can't be undone but conjectures can never achieve such a status.
– Geoff Robinson May 7, 2014 at 14:18

Finite reductive groups

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Weyl groups : $\underset{\mathfrak{S}_{n+1}}{\underset{\shortparallel}{A_n}}$, $\underset{(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n}{\underset{\shortparallel}{B_n}} \cong C_n$, $\underset{(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_n}{\underset{\shortparallel}{D_n}}$, E_6, E_7, E_8, F_4, G_2

Real and Complex Reflection Groups

V $\mathbb{C}$-vector space , $\dim_{\mathbb{C}} V = n < \infty$

$s \in GL(V)$ is a **pseudo-reflection** if

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If W is an irreducible complex reflection group, then

- either $W \cong G(de,e,n)$, the group of $n \times n$ monomial matrices with non-zero entries that are de-th roots of unity whose product is a d-th root of unity
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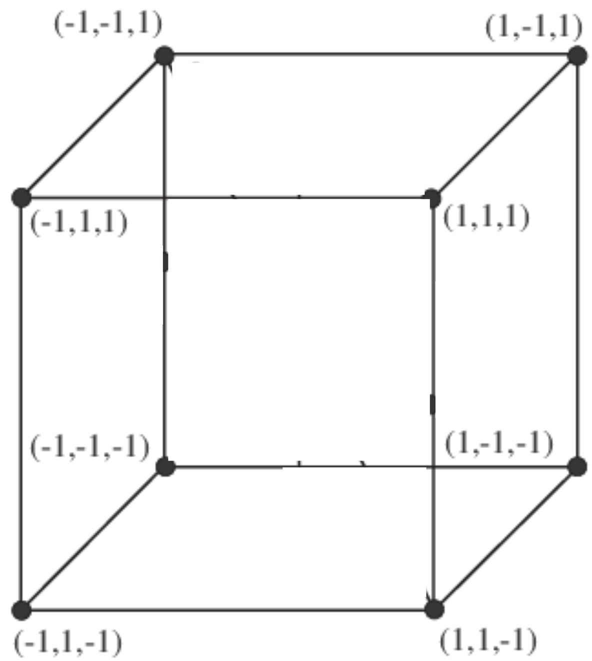
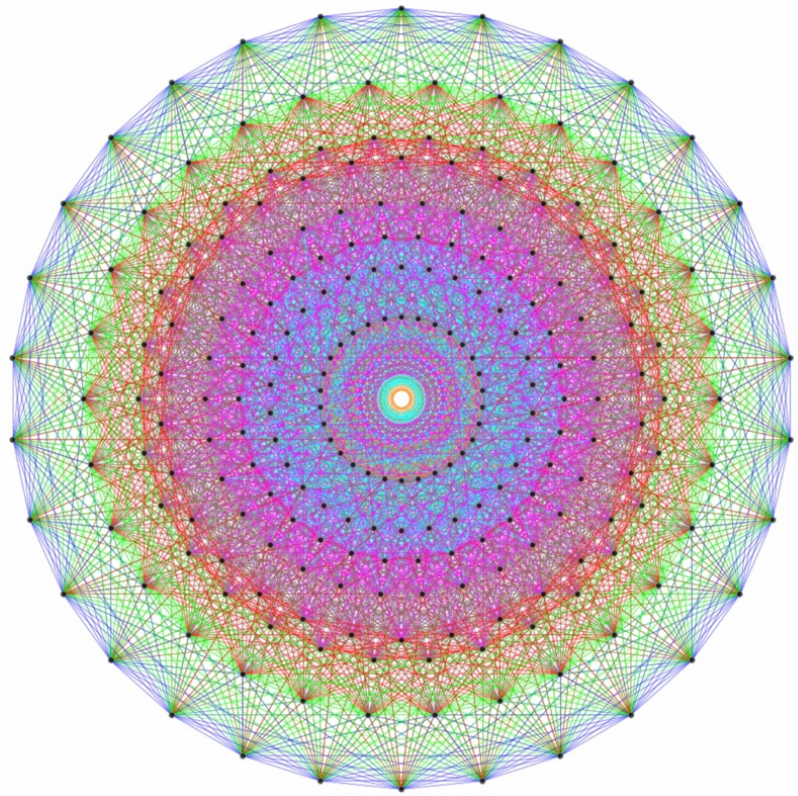


Symmetry group
$G(2,1,3)$

Allowing d-th roots of unity instead of square roots would create a "cube" in $\mathbb{C}^3$ with $G(d,1,3)$ as its symmetry group.



"Symmetry group" $E_8 = G_{37}$

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- G_9 by Chavli & Pfeiffer

Other applications

- Symplectic resolutions & Cherednik algebras $\rightsquigarrow$ Calogero-Moser spaces

[V/W smooth $\Leftrightarrow$ W complex reflection group
$W \curvearrowright V \oplus V^*$ is a symplectic reflection group]

- Knot invariants

[$\mathcal{H}_q(\mathfrak{S}_n) \rightsquigarrow$ Jones polynomial, HOMFLYPT polynomial
$\mathcal{H}_q(G(d,1,n)) \rightsquigarrow$ Geck-Lambropoulou invariants in the solid torus]

- Quantum groups & KLR algebras

- Theoretical physics

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