Generalizing the Temperley-Lieb algebra

Maria Chlouveraki

National and Kapodistrian University of Athens

Generalizing the Temperley-Lieb algebra

Maria Chlouveraki

National and Kapodistrian University of Athens

Joint work with Guillaume Pouchin

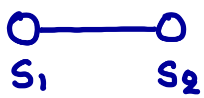
The real reflection group $\mathfrak{S}_n$

$$\mathfrak{S}_n = \left\langle s_1, s_2, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & i = 1, \dots, n-2 \\ s_i s_j = s_j s_i & \text{av } |i-j| > 1 \\ s_i^2 = 1 & \end{array} \right\rangle \qquad s_i = (i, i+1)$$

The real reflection group $\mathfrak{S}_n$

$$\mathfrak{S}_n = \left\langle s_1, s_2, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & i = 1, \dots, n-2 \\ s_i s_j = s_j s_i & \text{av } |i-j| > 1 \\ s_i^2 = 1 & \end{array} \right\rangle \qquad s_i = (i, i+1)$$

Dynkin diagram :

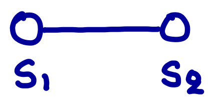


$$s_1 s_2 s_1 = s_2 s_1 s_2$$

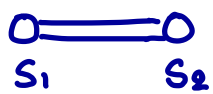
The real reflection group $\mathfrak{S}_n$

$$\mathfrak{S}_n = \left\langle s_1, s_2, \ldots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & i = 1, \ldots, n-2 \\ s_i s_j = s_j s_i & \text{av } |i-j| > 1 \\ s_i^2 = 1 & \end{array} \right\rangle \qquad s_i = (i, i+1)$$

Dynkin diagram :



$s_1 s_2 s_1 = s_2 s_1 s_2$

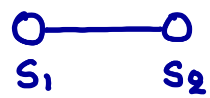


$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$

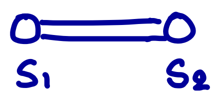
The real reflection group $\mathfrak{S}_n$

$$\mathfrak{S}_n = \left\langle s_1, s_2, \ldots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & i = 1, \ldots, n-2 \\ s_i s_j = s_j s_i & \text{av } |i-j| > 1 \\ s_i^2 = 1 & \end{array} \right\rangle \qquad s_i = (i, i+1)$$

Dynkin diagram :



$s_1 s_2 s_1 = s_2 s_1 s_2$



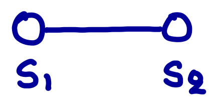
$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$

$s_1 s_2 = s_2 s_1$

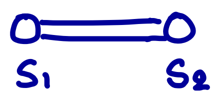
The real reflection group $\mathfrak{S}_n$

$$\mathfrak{S}_n = \left\langle s_1, s_2, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & i = 1, \dots, n-2 \\ s_i s_j = s_j s_i & \text{av } |i-j| > 1 \\ s_i^2 = 1 & \end{array} \right\rangle \qquad s_i = (i, i+1)$$

Dynkin diagram :



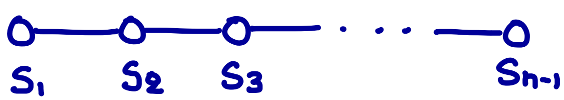
$s_1 s_2 s_1 = s_2 s_1 s_2$



$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$

$s_1 s_2 = s_2 s_1$

$\mathfrak{S}_n$

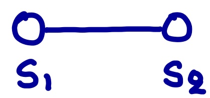


$s_i^2 = 1$

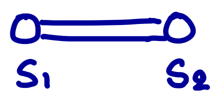
The real reflection group $\mathfrak{S}_n$

$$\mathfrak{S}_n = \left\langle s_1, s_2, \ldots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & i = 1, \ldots, n-2 \\ s_i s_j = s_j s_i & \text{av } |i-j| > 1 \\ s_i^2 = 1 & \end{array} \right\rangle \qquad s_i = (i, i+1)$$

Dynkin diagram :



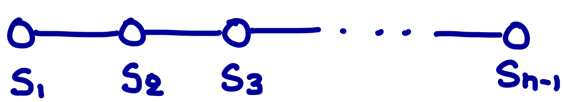
$s_1 s_2 s_1 = s_2 s_1 s_2$



$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$

$s_1 s_2 = s_2 s_1$

$\mathfrak{S}_n$



$s_i^2 = 1$

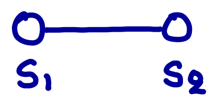
Type A_{n-1}

The Iwahori-Hecke algebra of type A [Iwahori 1964]

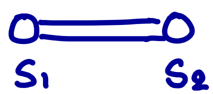
$$\mathfrak{S}_n = \left\langle s_1, s_2, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & i = 1, \dots, n-2 \\ s_i s_j = s_j s_i & \text{av } |i-j| > 1 \\ s_i^2 = 1 & \end{array} \right\rangle$$

$s_i = (i, i+1)$

Dynkin diagram :



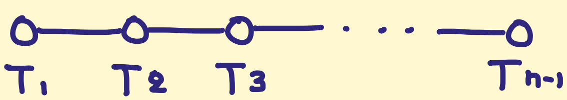
$s_1 s_2 s_1 = s_2 s_1 s_2$



$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$

$s_1 s_2 = s_2 s_1$

$\mathcal{H}_n(q)$



$T_i^2 = q + (q-1) T_i$ $\quad$ $\mathbb{C}(q)$-algebra

Representations of $\mathfrak{S}_n$ and $\mathcal{H}_n(q)$

$$\mathrm{Irr}(\mathfrak{S}_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_n(q)) \longleftrightarrow \{\text{ partitions of } n\} = \mathcal{P}(n)$$

$$\rho^\lambda \longleftarrow\!\shortmid \lambda$$

Representations of $\mathfrak{S}_n$ and $\mathcal{H}_n(q)$

$$\mathrm{Irr}(\mathfrak{S}_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_n(q)) \longleftrightarrow \{\text{partitions of } n\} = \mathcal{P}(n)$$

$$\rho^\lambda \longleftarrow\!\shortmid \lambda$$

<u>Example</u> : $n = 3$

(3) $(2,1)$ $(1,1,1)$ $\longleftarrow \lambda$

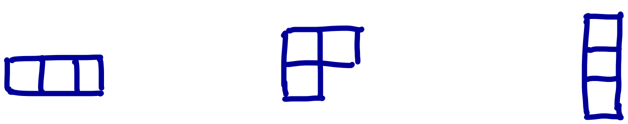
Representations of $\mathfrak{S}_n$ and $\mathcal{H}_n(q)$

$$\mathrm{Irr}(\mathfrak{S}_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_n(q)) \longleftrightarrow \{\text{partitions of } n\} = \mathcal{P}(n)$$

$$\rho^\lambda \longleftarrow \lambda$$

<u>Example</u> : $n = 3$

$(3) \qquad (2,1) \qquad (1,1,1) \qquad \longleftarrow \lambda$



$\longleftarrow Y(\lambda)$

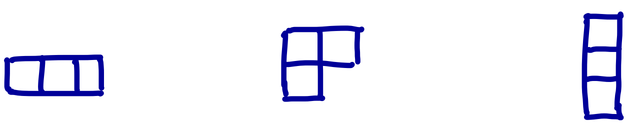
Representations of $\mathfrak{S}_n$ and $\mathcal{H}_n(q)$

$$\mathrm{Irr}(\mathfrak{S}_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_n(q)) \longleftrightarrow \{\text{partitions of } n\} = \mathcal{P}(n)$$

$$\rho^\lambda \longleftarrow \lambda$$

<u>Example</u> : $n = 3$

(3)	$(2,1)$	$(1,1,1)$	$\longleftarrow \lambda$



$\longleftarrow Y(\lambda)$

3	2	1	$\longleftarrow c(\lambda) = \#$ columns of $Y(\lambda)$

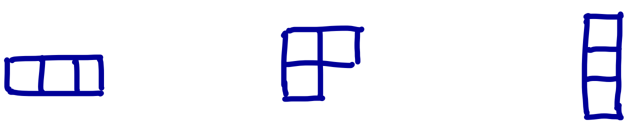
Representations of $\mathfrak{S}_n$ and $\mathcal{H}_n(q)$

$$\mathrm{Irr}(\mathfrak{S}_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_n(q)) \longleftrightarrow \{\text{ partitions of } n \} = \mathcal{P}(n)$$

$$\rho^\lambda \longleftarrow \lambda$$

<u>Example</u> : $n = 3$

(3) $\quad$ $(2,1)$ $\quad$ $(1,1,1)$ $\quad \longleftarrow \lambda$



$\longleftarrow Y(\lambda)$

3 $\quad$ 2 $\quad$ 1 $\quad \longleftarrow c(\lambda) = \#$ columns of $Y(\lambda)$

The Temperley-Lieb algebra [Temperley-Lieb, 1971]

$$TL_n(q) := \mathcal{H}_q(\mathfrak{S}_n) \big/ \big\langle T_{i,i+1} \mid i = 1, \dots, n-2 \big\rangle$$

where $T_{i,i+1} := 1 + T_i + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i$.

The Temperley-Lieb algebra [Temperley-Lieb, 1971]

$$TL_n(q) := \mathcal{H}_q(\mathfrak{S}_n) / \langle T_{i,i+1} \mid i = 1, \ldots, n-2 \rangle$$

where $T_{i,i+1} := 1 + T_i + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i$.

$$\boxed{\langle T_{i,i+1} \mid i = 1, \ldots, n-2 \rangle = \langle T_{1,2} \rangle \quad \text{principal ideal}}$$

The Temperley-Lieb algebra [Temperley-Lieb, 1971]

$$TL_n(q) := \mathcal{H}_q(\mathfrak{S}_n) / \langle T_{i,i+1} \mid i = 1, \dots, n-2 \rangle$$

where $T_{i,i+1} := 1 + T_i + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i$.

$$\langle T_{i,i+1} \mid i = 1, \dots, n-2 \rangle = \langle T_{1,2} \rangle \quad \text{principal ideal}$$

$$\mathrm{Irr}(TL_n(q)) \longleftrightarrow \{ \rho^\lambda \in \mathrm{Irr}(\mathcal{H}_n(q)) \mid \rho^\lambda(T_{1,2}) = 0 \}$$

The Temperley-Lieb algebra [Temperley-Lieb, 1971]

$$TL_n(q) := \mathcal{H}_q(\mathfrak{S}_n) \big/ \langle T_{i,i+1} \mid i = 1, \dots, n-2 \rangle$$

where $T_{i,i+1} := 1 + T_i + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i$.

$$\boxed{\langle T_{i,i+1} \mid i = 1, \dots, n-2 \rangle = \langle T_{1,2} \rangle \quad \text{principal ideal}}$$

$$\mathrm{Irr}(TL_n(q)) \longleftrightarrow \{ \rho^\lambda \in \mathrm{Irr}(\mathcal{H}_n(q)) \mid \rho^\lambda(T_{1,2}) = 0 \}$$

$$\|$$

$$\{ \rho^\lambda \in \mathrm{Irr}(\mathcal{H}_n(q)) \mid \mathrm{Res}^{\mathcal{H}_n(q)}_{\mathcal{H}_3(q)}(\rho^\lambda)(T_{1,2}) = 0 \}$$

The Temperley-Lieb algebra [Temperley-Lieb, 1971]

$$TL_n(q) := \mathcal{H}_q(\mathfrak{S}_n) / \langle T_{i,i+1} \mid i = 1, \ldots, n-2 \rangle$$

where $T_{i,i+1} := 1 + T_i + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i$.

$\langle T_{i,i+1} \mid i = 1, \ldots, n-2 \rangle = \langle T_{1,2} \rangle$ principal ideal

$$\mathrm{Irr}(TL_n(q)) \longleftrightarrow \{ \rho^\lambda \in \mathrm{Irr}(\mathcal{H}_n(q)) \mid \rho^\lambda(T_{1,2}) = 0 \}$$

$$\updownarrow \qquad\qquad\qquad\qquad \|$$

$$\{ \lambda \in \mathcal{P}(n) \mid c(\lambda) \leq 2 \} \longleftrightarrow \{ \rho^\lambda \in \mathrm{Irr}(\mathcal{H}_n(q)) \mid \mathrm{Res}^{\mathcal{H}_n(q)}_{\mathcal{H}_3(q)}(\rho^\lambda)(T_{1,2}) = 0 \}$$

Branching rule for $\mathfrak{S}_n$ (and $\mathcal{H}_n(q)$)

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \rho^{\lambda} = \bigoplus \rho^{Y(\lambda) \setminus \square}$$

Branching rule for $\mathfrak{S}_n$ (and $\mathcal{H}_n(q)$)

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \rho^{\lambda} = \bigoplus \rho^{Y(\lambda)\setminus\square}$$

<u>Example:</u> $$\mathrm{Res}^{\mathcal{H}_5(q)}_{\mathcal{H}_4(q)} \rho^{(3,1,1)} = \rho^{(2,1,1)} \oplus \rho^{(3,1)}$$

Branching rule for $\mathfrak{S}_n$ (and $\mathcal{H}_n(q)$)

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \rho^{\lambda} = \bigoplus \rho^{Y(\lambda)\setminus\square}$$

<u>Example:</u>

$$\operatorname{Res}^{\mathcal{H}_5(q)}_{\mathcal{H}_4(q)} \rho^{\ydiagram{3,1,1}} = \rho^{\ydiagram{2,1,1}} \oplus \rho^{\ydiagram{3,1}}$$

$$\operatorname{Res}^{\mathcal{H}_5(q)}_{\mathcal{H}_3(q)} \rho^{\ydiagram{3,1,1}} = \rho^{\ydiagram{1,1,1}} \oplus \rho^{\ydiagram{2,1}} \oplus \rho^{\ydiagram{2,1}} \oplus \rho^{\ydiagram{3}}$$

Branching rule for $\mathfrak{S}_n$ (and $\mathcal{H}_n(q)$)

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \rho^{\lambda} = \bigoplus \rho^{Y(\lambda)\setminus\square}$$

<u>Example:</u>

$$\mathrm{Res}^{\mathcal{H}_5(q)}_{\mathcal{H}_4(q)} \rho^{(3,1,1)} = \rho^{(2,1,1)} \oplus \rho^{(3,1)}$$

$$\mathrm{Res}^{\mathcal{H}_5(q)}_{\mathcal{H}_3(q)} \rho^{(3,1,1)} = \rho^{(1,1,1)} \oplus \rho^{(2,1)} \oplus \rho^{(2,1)} \oplus \rho^{(3)}$$

$$\boxed{\rho^{(1,1,1)}(T_{1,2}) = \rho^{(2,1)}(T_{1,2}) = 0 \neq \rho^{(3)}(T_{1,2})}$$

Branching rule for $\mathfrak{S}_n$ (and $\mathcal{H}_n(q)$)

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \rho^{\lambda} = \bigoplus \rho^{Y(\lambda)\setminus \square}$$

<u>Example:</u>

$$\mathrm{Res}^{\mathcal{H}_5(q)}_{\mathcal{H}_4(q)} \rho^{(3,1,1)} = \rho^{(2,1,1)} \oplus \rho^{(3,1)}$$

$$\mathrm{Res}^{\mathcal{H}_5(q)}_{\mathcal{H}_3(q)} \rho^{(3,1,1)} = \rho^{(1,1,1)} \oplus \rho^{(2,1)} \oplus \rho^{(2,1)} \oplus \rho^{(3)}$$

$$\rho^{(1,1,1)}(T_{1,2}) = \rho^{(2,1)}(T_{1,2}) = 0 \neq \rho^{(3)}(T_{1,2})$$

POLYNOMIAL KNOT INVARIANTS

Markov trace on $TL_n(q) \rightsquigarrow$ Jones polynomial [1984]

$\mathcal{H}_n(q) \rightsquigarrow$ HOMFLY-PT polynomial [1985]

The real reflection group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$

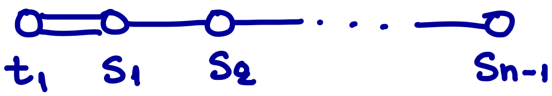
The complex reflection group $(\mathbb{Z}/d\mathbb{Z})^n \rtimes \mathfrak{S}_n$

The complex reflection group $(\mathbb{Z}/d\mathbb{Z})^n \rtimes \mathfrak{S}_n =: G(d,1,n)$

The real reflection group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n =: G(2,1,n)$

The real reflection group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n =: B_n$

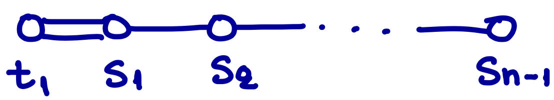
B_n



$s_i^2 = t_1^2 = 1$

The real reflection group $\underline{(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n} =: B_n$

B_n

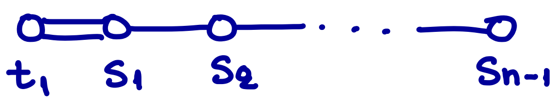


$s_i^2 = t_1^2 = 1$

$t_{j+1} = s_j t_j s_j^{-1}$

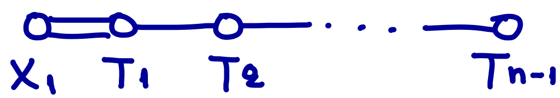
The Iwahori-Hecke algebra of type B [Iwahori, 1964]

B_n



$s_i^2 = t_1^2 = 1$ $\qquad$ $t_{j+1} = s_j t_j s_j^{-1}$

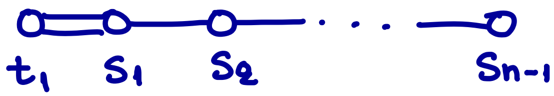
$\mathcal{H}_{2,n}(q)$



$T_i^2 = q + (q-1)T_i \;, \quad X_1^2 = q + (q-1)X_1$

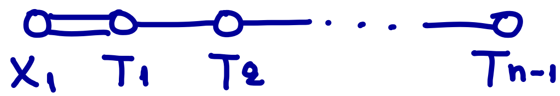
The Hecke algebra of type $G(d,1,n)$ [Ariki-Koike '94, Broué-Malle '93]

B_n



$s_i^2 = t_1^2 = 1$ $\qquad t_{j+1} = s_j t_j s_j^{-1}$

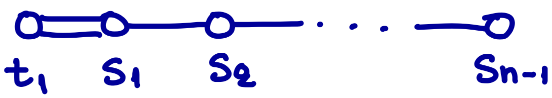
$\mathcal{H}_{2,n}(q)$



$T_i^2 = q + (q-1)T_i \ , \quad X_1^2 = q + (q-1)X_1$

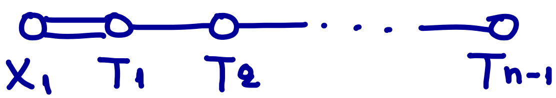
The Hecke algebra of type $G(d,1,n)$ [Ariki-Koike '94, Broué-Malle '93]

B_n



$s_i^2 = t_1^2 = 1$ $\qquad$ $t_{j+1} = s_j t_j s_j^{-1}$

$\mathcal{H}_{2,n}(q)$



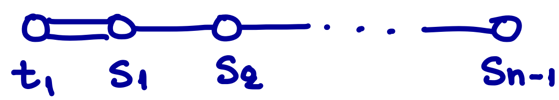
$T_i^2 = q + (q-1)T_i \quad , \quad X_1^2 = q + (q-1)X_1$

Markov trace on $\mathcal{H}_{2,n}(q) \rightsquigarrow$ Geck-Lambropoulou solid torus invariants [1997]

$\mathcal{H}_{d,n}(q) \rightsquigarrow$ Lambropoulou solid torus invariants [1999]

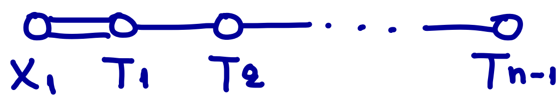
The Yokonuma-Hecke algebra of type A [Yokonuma, 1967]

B_n



$s_i^2 = t_1^2 = 1$ $\qquad t_{j+1} = s_j t_j s_j^{-1}$

$\mathcal{H}_{2,n}(q)$

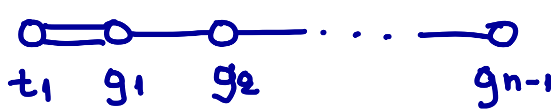


$T_i^2 = q + (q-1)T_i \,, \quad X_1^2 = q + (q-1)X_1$

Markov trace on $\mathcal{H}_{2,n}(q) \rightsquigarrow$ Geck-Lambropoulou solid torus invariants [1997]

$\mathcal{H}_{d,n}(q) \rightsquigarrow$ Lambropoulou solid torus invariants [1999]

$Y_{2,n}(q)$

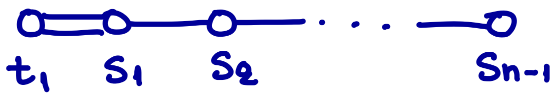


$g_i^2 = q + (q-1)e_i g_i \,, \quad t_1^2 = 1$

$$e_i = \frac{1 + t_i t_{i+1}}{2} \qquad t_{j+1} = g_j t_j g_j^{-1}$$

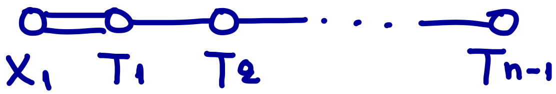
The Yokonuma-Hecke algebra of type A [Yokonuma, 1967]

B_n



$s_i^2 = t_1^2 = 1$ $\qquad t_{j+1} = s_j t_j s_j^{-1}$

$\mathcal{H}_{2,n}(q)$

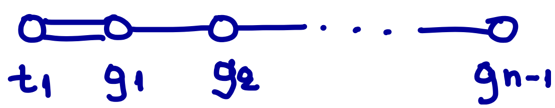


$T_i^2 = q + (q-1)T_i \,, \quad X_1^2 = q + (q-1)X_1$

Markov trace on $\mathcal{H}_{2,n}(q) \rightsquigarrow$ Geck-Lambropoulou solid torus invariants [1997]

$\mathcal{H}_{d,n}(q) \rightsquigarrow$ Lambropoulou solid torus invariants [1999]

$Y_{2,n}(q)$



$g_i^2 = q + (q-1)e_i g_i \,, \quad t_1^2 = 1$

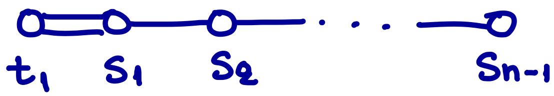
$e_i = \dfrac{1 + t_i t_{i+1}}{2}$ $\qquad t_{j+1} = g_j t_j g_j^{-1}$

Markov trace on $Y_{d,n}(q) \rightsquigarrow \Theta$ - invariant

[Juyumaya-Lambropoulou '13, C.-Juyumaya-Karvounis-Lambropoulou '20]

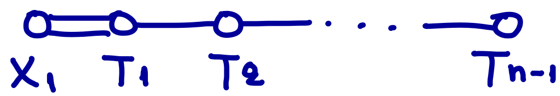
The Yokonuma-Hecke algebra of type A [Yokonuma, 1967]

B_n



$s_i^2 = t_1^2 = 1$ $\qquad t_{j+1} = s_j t_j s_j^{-1}$

$\mathcal{H}_{2,n}(q)$

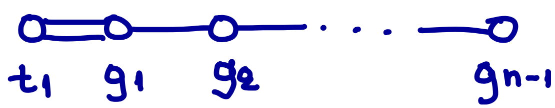


$T_i^2 = q + (q-1)T_i \,, \quad X_1^2 = q + (q-1)X_1$

Markov trace on $\mathcal{H}_{2,n}(q) \rightsquigarrow$ Geck-Lambropoulou solid torus invariants [1997]

$\mathcal{H}_{d,n}(q) \rightsquigarrow$ Lambropoulou solid torus invariants [1999]

$Y_{2,n}(q)$



$g_i^2 = q + (q-1)e_i g_i \,, \quad t_1^2 = 1$

$e_i = \dfrac{1 + t_i t_{i+1}}{2}$ $\qquad t_{j+1} = g_j t_j g_j^{-1}$

Markov trace on $Y_{d,n}(q) \rightsquigarrow \Theta$-invariant stronger than HOMFLYPT

[Juyumaya-Lambropoulou '13, C.-Juyumaya-Karvounis-Lambropoulou '20]

$$\mathrm{Irr}(B_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_{2,n}(q)) \longleftrightarrow \mathrm{Irr}(Y_{2,n}(q)) \longleftrightarrow \{2\text{-partitions of } n\} =: \mathcal{P}_2(n)$$

$$\rho^\lambda \longleftrightarrow \lambda = (\lambda^1, \lambda^2)$$

$$\mathrm{Irr}(B_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_{2,n}(q)) \longleftrightarrow \mathrm{Irr}(Y_{2,n}(q)) \longleftrightarrow \{2\text{-partitions of } n\} =: \mathcal{P}_2(n)$$

$$\rho^\lambda \longleftrightarrow \lambda = (\lambda^1, \lambda^2)$$

<u>Example</u> : $n = 3$

$$\left(\ydiagram{3}, \emptyset\right), \left(\ydiagram{2,1}, \emptyset\right), \left(\ydiagram{1,1,1}, \emptyset\right),$$

$$\mathrm{Irr}(B_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_{2,n}(q)) \longleftrightarrow \mathrm{Irr}(Y_{2,n}(q)) \longleftrightarrow \{2\text{-partitions of } n\} =: \mathcal{P}_2(n)$$

$$\rho^{\lambda} \longleftrightarrow \lambda = (\lambda^1, \lambda^2)$$

<u>Example</u> : $n = 3$

$$(\ydiagram{3}, \emptyset), \quad (\ydiagram{2,1}, \emptyset), \quad (\ydiagram{1,1,1}, \emptyset),$$

$$(\emptyset, \ydiagram{3}), \quad (\emptyset, \ydiagram{2,1}), \quad (\emptyset, \ydiagram{1,1,1}),$$

$$\mathrm{Irr}(B_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_{2,n}(q)) \longleftrightarrow \mathrm{Irr}(Y_{2,n}(q)) \longleftrightarrow \{2\text{-partitions of } n\} =: \mathcal{P}_2(n)$$

$$\rho^{\lambda} \longleftrightarrow \lambda = (\lambda^1, \lambda^2)$$

<u>Example</u> : $n = 3$

$(\ydiagram{3}, \emptyset)$, $(\ydiagram{2,1}, \emptyset)$, $(\ydiagram{1,1,1}, \emptyset)$,

$(\emptyset, \ydiagram{3})$, $(\emptyset, \ydiagram{2,1})$, $(\emptyset, \ydiagram{1,1,1})$,

$(\ydiagram{2}, \ydiagram{1})$, $(\ydiagram{1,1}, \ydiagram{1})$,

$$\mathrm{Irr}(B_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_{2,n}(q)) \longleftrightarrow \mathrm{Irr}(Y_{2,n}(q)) \longleftrightarrow \{2\text{-partitions of } n\} =: \mathcal{P}_2(n)$$

$$\rho^{\lambda} \longmapsfrom \lambda = (\lambda^1, \lambda^2)$$

<u>Example</u> : $n = 3$

$$((3), \emptyset),\ ((2,1), \emptyset),\ ((1,1,1), \emptyset),$$

$$(\emptyset, (3)),\ (\emptyset, (2,1)),\ (\emptyset, (1,1,1)),$$

$$((2), (1)),\ ((1,1), (1)),$$

$$((1), (2)),\ ((2), (1)).$$

$$\mathrm{Irr}(B_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_{2,n}(q)) \longleftrightarrow \mathrm{Irr}(Y_{2,n}(q)) \longleftrightarrow \{2\text{-partitions of } n\} =: \mathcal{P}_2(n)$$

$$\rho^{\lambda} \quad \longleftrightarrow \quad \lambda = (\lambda^1, \lambda^2)$$

<u>Example</u> : $n = 3$

$$((3), \emptyset), \quad ((2,1), \emptyset), \quad ((1,1,1), \emptyset),$$

$$(\emptyset, (3)), \quad (\emptyset, (2,1)), \quad (\emptyset, (1,1,1)),$$

$$((2), (1)), \quad ((1,1), (1)),$$

$$((1), (2)), \quad ((2), (1)).$$

$$\mathcal{H}_{2,n}(q) \twoheadrightarrow TL_{2,n}(q) \twoheadrightarrow \text{Blob algebra}$$

[Graham'85] [Martin-Saleur'94]

$$\mathrm{Irr}(B_n) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_{2,n}(q)) \longleftrightarrow \mathrm{Irr}(Y_{2,n}(q)) \longleftrightarrow \{2\text{-partitions of } n\} =: \mathcal{P}_2(n)$$

$$\rho^\lambda \longmapsfrom \lambda = (\lambda^1, \lambda^2)$$

<u>Example</u> : $n = 3$

$(\ydiagram{3}, \emptyset), \ (\ydiagram{2,1}, \emptyset), \ (\ydiagram{1,1,1}, \emptyset),$

$(\emptyset, \ydiagram{3}), \ (\emptyset, \ydiagram{2,1}), \ (\emptyset, \ydiagram{1,1,1}),$

$(\ydiagram{2}, \ydiagram{1}), \ (\ydiagram{1,1}, \ydiagram{1}),$

$(\ydiagram{1}, \ydiagram{2}), \ (\ydiagram{2}, \ydiagram{1}).$

$\mathcal{H}_{2,n}(q) \twoheadrightarrow TL_{2,n}(q) \twoheadrightarrow$ Blob algebra

[Graham'85] [Martin-Saleur'94]

$\mathcal{H}_{d,n}(q) \twoheadrightarrow TL_{d,n}(q)$

[Lehrer-Lyu'22]

The Yokonuma-Temperley-Lieb algebra [Goundaroulis-Juyumaya-Kontogeorgis-Lambropoulou, 2014]

$$g_{1,2} := 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1$$

$$YTL_{2,n}(q) := Y_{2,n}(q) / \langle g_{1,2} \rangle$$

The Yokonuma-Temperley-Lieb algebra [Goundaroulis-Juyumaya-Kontogeorgis-Lambropoulou, 2014]

$$g_{1,2} := 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1$$

$$YTL_{2,n}(q) := Y_{2,n}(q) / \langle g_{1,2} \rangle$$

$$\mathrm{Irr}(YTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in P_2(n) \mid \mathrm{Res}^{B_n}_{\mathfrak{S}_3}(\rho^\lambda)(g_{1,2}) = 0 \}$$

The Yokonuma-Temperley-Lieb algebra [Goundaroulis-Juyumaya-Kontogeorgis-Lambropoulou, 2014]

$$g_{1,2} := 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1$$

$$YTL_{2,n}(q) := Y_{2,n}(q) / \langle g_{1,2} \rangle$$

$$\operatorname{Irr}(YTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in P_2(n) \mid \operatorname{Res}^{B_n}_{\mathfrak{S}_3}(\rho^\lambda)(g_{1,2}) = 0 \}$$

$$\operatorname{Res}^{B_n}_{\mathfrak{S}_n}(\rho^\lambda) = \sum c^{\mu}_{\lambda^1,\lambda^2}\, \mu$$

Littlewood-Richardson coefficients

The Yokonuma-Temperley-Lieb algebra [Goundaroulis-Juyumaya-Kontogeorgis-Lambropoulou, 2014]

$$g_{1,2} := 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1$$

$$YTL_{2,n}(q) := Y_{2,n}(q) / \langle g_{1,2} \rangle$$

$$\mathrm{Irr}(YTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in \mathcal{P}_2(n) \mid \mathrm{Res}^{B_n}_{\mathfrak{S}_3}(\rho^\lambda)(g_{1,2}) = 0 \}$$

$$\mathrm{Res}^{B_n}_{\mathfrak{S}_n}(\rho^\lambda) = \sum c^{\mu}_{\lambda^1,\lambda^2}\, \mu$$

Littlewood-Richardson coefficients

<u>Theorem</u> [C.-Pouchin '15] $\mathrm{Irr}(YTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in \mathcal{P}_2(n) \mid c(\lambda_1) + c(\lambda_2) \leq 2 \}$

The Yokonuma-Temperley-Lieb algebra [Goundaroulis-Juyumaya-Kontogeorgis-Lambropoulou, 2014]

$$g_{1,2} := 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1$$

$$YTL_{2,n}(q) := Y_{2,n}(q) / \langle g_{1,2} \rangle$$

$$\mathrm{Irr}(YTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in \mathcal{P}_2(n) \mid \mathrm{Res}^{B_n}_{\mathfrak{S}_3}(\rho^\lambda)(g_{1,2}) = 0 \}$$

$$\mathrm{Res}^{B_n}_{\mathfrak{S}_n}(\rho^\lambda) = \sum c^{\mu}_{\lambda^1,\lambda^2}\, \mu$$

Littlewood-Richardson coefficients

Theorem [C.-Pouchin '15] $\mathrm{Irr}(YTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in \mathcal{P}_2(n) \mid c(\lambda^1) + c(\lambda^2) \leq 2 \}$

Example: $(\ydiagram{1,1}, \ydiagram{1})$ ✓ $\qquad (\ydiagram{2}, \ydiagram{1})$ ✗ $\qquad (\emptyset, \ydiagram{3})$ ✗

The Yokonuma-Temperley-Lieb algebra [Goundaroulis-Juyumaya-Kontogeorgis-Lambropoulou, 2014]

$$g_{1,2} := 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1$$

$$YTL_{2,n}(q) := Y_{2,n}(q) / \langle g_{1,2} \rangle$$

$$\mathrm{Irr}(YTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in \mathcal{P}_2(n) \mid \mathrm{Res}^{B_n}_{\mathfrak{S}_3}(\rho^\lambda)(g_{1,2}) = 0 \}$$

$$\mathrm{Res}^{B_n}_{\mathfrak{S}_n}(\rho^\lambda) = \sum c^{\mu}_{\lambda^1,\lambda^2}\, \mu$$

Littlewood-Richardson coefficients

Theorem [C.-Pouchin '15] $\mathrm{Irr}(YTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in \mathcal{P}_2(n) \mid c(\lambda^1) + c(\lambda^2) \leq 2 \}$

Example: $\left(\ydiagram{1,1}\,,\ \ydiagram{1}\right)$ ✓ $\qquad \left(\ydiagram{2}\,,\ \ydiagram{1}\right)$ ✗ $\qquad \left(\emptyset\,,\ \ydiagram{3}\right)$ ✗

Markov trace on $YTL_{d,n}(q) \rightsquigarrow$ Jones polynomial

The Framization of the Temperley-Lieb algebra, and the Complex Reflection Temperley-Lieb algebra [GJKL, 2017]

$$FTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{\substack{0 \leq a,b,c \leq 1 \\ a+b+c \equiv 0 \bmod 2}} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

$$CTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{0 \leq a,b,c \leq 1} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

The Framization of the Temperley-Lieb algebra, and the Complex Reflection Temperley-Lieb algebra [GJKL, 2017]

$$FTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{\substack{0\leq a,b,c\leq 1 \\ a+b+c\equiv 0 \bmod 2}} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

$$CTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{0\leq a,b,c\leq 1} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

<u>Theorem</u> [C.-Pouchin '17] $\mathrm{Irr}(FTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in P_2(n) \mid c(\lambda^i) \leq 2 \ \forall i=1,2 \}$

The Framization of the Temperley-Lieb algebra, and the Complex Reflection Temperley-Lieb algebra [GJKL, 2017]

$$FTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{\substack{0 \leq a,b,c \leq 1 \\ a+b+c \equiv 0 \bmod 2}} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

$$CTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{0 \leq a,b,c \leq 1} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

<u>Theorem</u> [C.-Pouchin '17] $\mathrm{Irr}(FTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in \mathcal{P}_2(n) \mid c(\lambda^i) \leq 2 \ \forall\, i = 1,2 \}$

<u>Example</u>: $(\ydiagram{1,1}, \ydiagram{1})$ ✓ $(\ydiagram{2}, \ydiagram{1})$ ✓ $(\emptyset, \ydiagram{3})$ ✗

The Framization of the Temperley-Lieb algebra, and the Complex Reflection Temperley-Lieb algebra [GJKL, 2017]

$$FTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{\substack{0\le a,b,c \le 1 \\ a+b+c \equiv 0 \bmod 2}} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

$$CTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{0\le a,b,c \le 1} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

Theorem [C.-Pouchin '17] $\mathrm{Irr}(FTL_{2,n}(q)) \longleftrightarrow \{\lambda \in P_2(n) \mid c(\lambda^i) \le 2 \ \forall i = 1,2\}$

Example: $\left(\ydiagram{1,1}\,,\ \ydiagram{1}\right)$ ✓ $\qquad \left(\ydiagram{2}\,,\ \ydiagram{1}\right)$ ✓ $\qquad \left(\emptyset\,,\ \ydiagram{3}\right)$ ✗

Theorem [C.-Pouchin '17] $\mathrm{Irr}(CTL_{2,n}(q)) \longleftrightarrow \{\lambda \in P_2(n) \mid c(\lambda_1) \le 2\}$

The Framization of the Temperley-Lieb algebra, and the Complex Reflection Temperley-Lieb algebra [GJKL, 2017]

$$FTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{\substack{0 \leq a,b,c \leq 1 \\ a+b+c \equiv 0 \bmod 2}} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

$$CTL_{2,n}(q) := Y_{2,n}(q) \Big/ \Big\langle \sum_{0 \leq a,b,c \leq 1} t_1^a t_2^b t_3^c \, g_{1,2} \Big\rangle$$

Theorem [C.-Pouchin '17] $\mathrm{Irr}(FTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in P_2(n) \mid c(\lambda^i) \leq 2 \;\; \forall\, i = 1, 2 \}$

Example: $(\ydiagram{1,1}\,,\ \ydiagram{1})$ ✓ $(\ydiagram{2}\,,\ \ydiagram{1})$ ✓ $(\emptyset\,,\ \ydiagram{3})$ ✗

Theorem [C.-Pouchin '17] $\mathrm{Irr}(CTL_{2,n}(q)) \longleftrightarrow \{ \lambda \in P_2(n) \mid c(\lambda_1) \leq 2 \}$

Example: $(\ydiagram{1,1}\,,\ \ydiagram{1})$ ✓ $(\ydiagram{2}\,,\ \ydiagram{1})$ ✓ $(\emptyset\,,\ \ydiagram{3})$ ✓

Theorem [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(\mathcal{H}_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

Theorem [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(\mathcal{H}_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

Example: $Y_{2,3}(q) \cong \mathcal{H}_3(q) \oplus \mathrm{Mat}_3(\mathcal{H}_2(q) \otimes \mathcal{H}_1(q))$

Theorem [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(\mathcal{H}_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

Example: $Y_{2,3}(q) \cong 2\mathcal{H}_3(q) \oplus 2\,\mathrm{Mat}_3\left(\mathcal{H}_2(q) \otimes \mathcal{H}_1(q)\right)$

Theorem [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(\mathcal{H}_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

Example: $Y_{2,3}(q) \cong 2\mathcal{H}_3(q) \oplus 2\,\mathrm{Mat}_3\left(\mathcal{H}_2(q) \otimes \cancel{\mathcal{H}_1(q)}\right)$

Theorem [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(\mathcal{H}_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

Example: $Y_{2,3}(q) \cong 2\mathcal{H}_3(q) \oplus 2\,\mathrm{Mat}_3\left(\mathcal{H}_2(q) \otimes \mathcal{H}_1(q)\right)$

Theorem [C.-Pouchin 2017]

$$FTL_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(TL_{n_1}(q) \otimes TL_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

Theorem [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(\mathcal{H}_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

Example: $Y_{2,3}(q) \cong 2\mathcal{H}_3(q) \oplus 2\,\mathrm{Mat}_3\left(\mathcal{H}_2(q) \otimes \mathcal{H}_1(q)\right)$

Theorem [C.-Pouchin 2017]

$$FTL_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(TL_{n_1}(q) \otimes TL_{n_2}(q)\right)$$

$$m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

$$CTL_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(TL_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right)$$

Theorem [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(\mathcal{H}_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

Example: $Y_{2,3}(q) \cong 2\mathcal{H}_3(q) \oplus 2\,\mathrm{Mat}_3\left(\mathcal{H}_2(q) \otimes \mathcal{H}_1(q)\right)$

Theorem [C.-Pouchin 2017]

$$FTL_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(TL_{n_1}(q) \otimes TL_{n_2}(q)\right)$$

$$m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

$$CTL_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(TL_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right)$$

Markov trace on $FTL_{d,n}(q) \rightsquigarrow \theta$-invariant

Theorem [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(\mathcal{H}_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

Example: $Y_{2,3}(q) \cong 2\mathcal{H}_3(q) \oplus 2\,\mathrm{Mat}_3(\mathcal{H}_2(q) \otimes \mathcal{H}_1(q))$

Theorem [C.-Pouchin 2017]

$$FTL_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(TL_{n_1}(q) \otimes TL_{n_2}(q)\right)$$

$$m_{n_1,n_2} = \frac{n!}{n_1!\, n_2!}$$

$$CTL_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(TL_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right)$$

Markov trace on $FTL_{d,n}(q) \rightsquigarrow \theta$-invariant stronger than Jones

Theorem [Lusztig 2005, Jacon-Poulain d'Andecy 2016]

$$Y_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(\mathcal{H}_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right) \qquad m_{n_1,n_2} = \frac{n!}{n_1!\,n_2!}$$

Example: $Y_{2,3}(q) \cong 2\mathcal{H}_3(q) \oplus 2\,\mathrm{Mat}_3(\mathcal{H}_2(q) \otimes \mathcal{H}_1(q))$

Theorem [C.-Pouchin 2017]

$$FTL_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(TL_{n_1}(q) \otimes TL_{n_2}(q)\right)$$

$$m_{n_1,n_2} = \frac{n!}{n_1!\,n_2!}$$

$$CTL_{2,n}(q) \cong \bigoplus_{n_1+n_2=n} \mathrm{Mat}_{m_{n_1,n_2}}\left(TL_{n_1}(q) \otimes \mathcal{H}_{n_2}(q)\right)$$

Markov trace on $FTL_{d,n}(q) \rightsquigarrow \theta$-invariant stronger than Jones

$CTL_{d,n}(q) \rightsquigarrow \theta$ or Θ

$$
\begin{tikzcd}
\mathrm{Y}_{d,n}(q) \arrow[r, two heads] \arrow[d, two heads, "t_j \mapsto 1"'] & \mathrm{CTL}_{d,n}(q) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{FTL}_{d,n}(q) \arrow[r, two heads] \arrow[dl, two heads] & \mathrm{YTL}_{d,n}(q) \arrow[dll, two heads] \\
\mathcal{H}_n(q) \arrow[r, two heads] & \mathrm{TL}_n(q) & &
\end{tikzcd}
$$

$$\begin{tikzcd}
Y_{d,n}(q) \arrow[r, two heads] \arrow[d, "t_j \mapsto 1"'] & CTL_{d,n}(q) \arrow[r, two heads] \arrow[d, two heads] & FTL_{d,n}(q) \arrow[r, two heads] \arrow[dl, two heads] & YTL_{d,n}(q) \arrow[dll, two heads] \\
\mathcal{H}_n(q) \arrow[r, two heads] & TL_n(q) & &
\end{tikzcd}$$